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NONPARAMETRIC SPECTRAL ANALYSIS ON DISJOINT SEGMENTS OF OBSERVATIONS

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Abstract

This paper presents a new approach to estimate the spectral density matrix of a strictly stationary r-vector valued time series in the case where there are some randomly missing observations. The modified series is defined and is used to construct the finite Fourier transform in L-disjoint segments of observations. The modified periodogram is defined and the spectral density estimate is constructed using it. Statistical properties, asymptotic moments and asymptotic normality of estimates are discussed. This method is applied to the northern hemisphere and southern hemisphere monthly temperature.

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1. INTRODUCTION

As early as the late 19 century Schuster (1898) introduced the periodogram which may be regarded as the origin of spectral analysis. In recent years there has been an increasing interest in spectral estimation using periodogram, Brillinger [(1969),(2001)], Brockwell and Davis (1991), Ghazal (2002), Ghazal and Farag (2001), Walden (2000) and Thomas (2001). The spectral analysis of time series with missing data is one of the most important problems faced by applied researchers whose data arise in the form of time series. Various researchers, including Jones (1962), Parzen [(1962), (1963)], Scheinok (1965), Bloomfield (1970) and Dahlhaus (1987) have discussed the problem of spectral density estimate in the case where there are missing data. Jones (1962) examined the case where a block of observations is periodically unobtainable. Parzen (1962) developed the theory of amplitude-modulated stationary processes, and applied this theory to missing data problems (Parzen(1963)), considering in detail the case where observations are missed in some periodic way. The amplitude-modulated series is constructed by replacing missing observations in the original series by their mean value. Scheinok (1965) considered the case where an observation is made or not according to the out come of a Bernoulli trial. Bloomfield (1970) considered the case where a more general random mechanism is involved, see also Ghazal et al (2002) and Ghazal and Elhassanein [(2006),(2007),(2008),(2009)]. In this paper we will construct an estimate of the spectral density matrix of a strictly stationary r-vector valued time series with randomly missing observations on disjoint segments of observations using periodogram, see Bartllet (1953) and Jankins and Watts (1968). Asymptotic properties and distribution will be derived. The northern hemisphere and southern hemisphere monthly temperature anomalies (degrees C) (1990-1999) are used to discuss theoretical results and to compare the classic results, in the case where all observations are available, with the case where there are some randomly missed observations. In the paper $W_r(\gamma, \sum)$ will denote an $r \times r$ symmetric matrix-valued Wishart variate with covariance matrix $\sum$ and γ degree of freedom. Let $W_r^c(\gamma, \sum)$ denote an $r \times r$ Hermitian matrix-valued complex Wishart variate with covariance matrix $\sum$ and γ degree of freedom. Let $N_r(\mu_Z, \sum_{ZZ})$ denote the multivariate normal distribution with mean $\mu_Z\{$ and covariance matrix $\sum_{ZZ}$ where Z is an r-vector valued random variable having real-valued components. Let $N_r^c(\mu_Z, \sum_{ZZ})$, the complex multivariate normal distribution with mean μ_Z and covariance matrix $\sum_{ZZ}$ where Z is of complex-valued components.

2. ORIGINAL SERIES

Let $X(t)(t = 0, \pm 1, ...)$ be a zero mean r-vector valued strictly stationary time series with

$$E\{X(t+u)\overline{X}'(t)\} = C_{XX}(u) \tag{1}$$

and

$$\sum_{u=-\infty}^{\infty} |C_{XX}(u)| \prec \infty, \tag{2}$$

where $|C_{XX}(u)|$ denotes the matrix of absolute values, the bar denotes the complex conjugate and $'{}'{}'$ denotes the matrix transpose. We may then define $f_{XX}(\lambda)$ the $r \times r$ matrix of second order spectral densities by

$$f_{XX}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} C_{XX}(u) \exp(-i\lambda u), \tag{3}$$

see Brillinger (2001) P.24.

Using the assumed stationary, we then set down

Assumption I. $X(t)$ is a strictly stationary series all of whose moments exist. For each $j = 1, 2, ..., k-1$ and any k-tuple $a_1, a_2, ..., a_k$ we have

$$\sum_{u_1 \dots u_{k-1}} |u_j C_{a_1,...,a_k}(u_1, ..., u_{k-1})| < \infty, k = 2, 3, ... \tag{4}$$

where

$$C_{a_1,...,a_k}(u_1, u_2, ..., u_{k-1}) = cum\{X_{a_1}(t+u_1), X_{a_2}(t+u_2), ..., X_{a_k}(t)\}, \tag{5}$$

$(a_1, a_2, ..., a_k = 1, 2, ..., r \ ; u_1, u_2, ..., u_{k-1}, t = 0, \pm 1, ...; k = 2, ...)$(see definition 2.3.1, in Brillinger (2001) pp. 19)

Because cumulants are measures of the joint dependance of random variables, (4) is seen to be a form of mixing or asymptotic independence requirement for values of $X(t)$ well separated in time. If $X(t)$ satisfies Assumption I we may define its cumulant spectral densities by

$$f_{a_1,...,a_k}(\lambda_1, ..., \lambda_{k-1}) = (2\pi)^{-k+1} \sum_{u_1...u_{k-1}} C_{a_1,...,a_k}(u_1, ..., u_{k-1}) \exp\Big(- i \sum_{j=1}^{k-1} \lambda_j u_j \Big), \tag{6}$$

$(-\infty < \lambda < \infty, a_1, a_2, ..., a_k = 1, 2, ..., r; k = 2, ...)$. If $k = 2$ the cross-spectra $f_{a_1 a_2}(\lambda)$ are collected together in the matrix $f_{XX}(\lambda)$ of (3).

Assumption II. $\Phi(t)$ is bounded, is of bounded variation and vanishes for $t < 0$, $t > T - 1$ that is called data window. Let

$$\Phi^{(T)}_{a_1...a_k}(\lambda) = \sum_t \prod_{j=1}^{k} \Phi_{a_j}(t) \exp(-i\lambda t), \tag{7}$$

for $-\infty < \lambda < \infty$ and $a_1, ..., a_k = 1, 2, ..., r$.

3. THE MODIFIED SERIES

Let $H(t) = \{H_a(t)(t = 0, \pm 1, ...)\}_{a=1,2,...,r}$ be a process independent of $X(t)$ such that, for every t

$$P\{H_a(t) = 1\} = p, P\{H_a(t) = 0\} = q, \tag{8}$$

note that

$$E\{H_a(t)\} = p. \tag{9}$$

The success of recording an observation not depend on the fail of another and so it is independent. We may then define the modified series

$$Y(t) = H(t)X(t), \tag{10}$$

with components,

$$Y_a(t) = H_a(t)X_a(t), \tag{11}$$

where

$$H(t) = \begin{cases} 1 & \text{if } X(t) \text{ is observied ;} \\ 0 & \text{if } X(t) \text{ is missed.} \end{cases} \tag{12}$$

4. FINITE FOURIER TRANSFORM IN L-DISJOINT SEGMENTS OF OBSERVATIONS

Let $X(t)(t = 0, 1, ..., T - 1)$ be an observed stretch of data with some randomly missing observations. Let $T = LN$, where L, is the number of disjoint segments of observations and N, is the length of each interval, then the observations may be represented as $X(lN), X(lN + 1), ..., X((l + 1)N - 1), l = 0, 1, ..., L - 1$.

The finite Fourier transform of a given stretch of data, is defined by

$$d_Y^{(T)}(\lambda) = \left(2\pi \sum_{t=lN}^{(l+1)N-1} \Phi^2(t-lN)\right)^{-\frac{1}{2}} \times \sum_{t=lN}^{(l+1)N-1} \Phi(t-lN)\exp(-i\lambda t)Y(t), \tag{13}$$

where $-\infty < \lambda < \infty$, and $\Phi(t)$ is the data window satisfies Assumption II.

Lemma 4.1. Let $\Phi_a(t)$ satisfy Assumption II and for $a = 1, 2, ..., r$, then

$$\sum_t \Phi_a(t+u)\Phi_b(t)\exp(-i\lambda t) = \sum_t \Phi_a(t)\Phi_b(t)\exp(-i\lambda t) + \epsilon \tag{14}$$

where $|\epsilon| < k\,|u|$, $u = 0, \pm 1, ...$

$$\left(2\pi \sum_{t=lN}^{(l+1)N-1} \Phi_a^2(t-lN)\right)^{-\frac{1}{2}} \times \left(2\pi \sum_{s=lN}^{(l+1)N-1} \Phi_b^2(s-lN)\right)^{-\frac{1}{2}} = O(N^{-1}) \tag{15}$$

$$\Phi_{a_1 \dots a_k}^{(lN)}(\lambda) = O(N) \tag{16}$$

where k is constant and $\Phi_{a_1.....a_k}^{(T)}(\lambda)$ is given by (7).

Proof. The proof of (14) is a special case of Lemma (P.4.1) in Brillinger (2001) PP. 420.

Let

$$\tau = \left(2\pi \sum_{t=lN}^{(l+1)N-1} \Phi^2(t-lN)\right)^{-\frac{1}{2}} \times \left(2\pi \sum_{s=lN}^{(l+1)N-1} \Phi^2(s-lN)\right)^{-\frac{1}{2}},$$

then

$$|\tau| = B\left|\sum_{t=lN}^{(l+1)N-1} \sum_{s=lN}^{(l+1)N-1} \Phi^2(t-lN)\Phi^2(s-lN)\right|^{-\frac{1}{2}}$$

$$\le B\left(\sum_{t=lN}^{(l+1)N-1} \sum_{s=lN}^{(l+1)N-1} \left|\Phi^2(t-lN)\Phi^2(s-lN)\right|\right)^{-\frac{1}{2}},$$

for constant B, by Assumption II the proof of (14) is completed. The proof of (16) is easy to derive. □

Theorem 4.1. Let$X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $d_a^{(lN)}(\lambda)$ be defined as (13), and $\Phi_a(t)$ satisfy Assumption II, for $a = 1, 2, ..., r$, then

$$E\{\, d_a^{(lN)}(\lambda)\} = 0 \tag{17}$$

$$Cov\{d_a^{(lN)}(\lambda_1), d_b^{(lN)}(-\lambda_2)\}$$

$$= p^2 R_{ab}\Phi_{ab}^{(lN)}(\lambda_1+\lambda_2) f_{ab}(\lambda_1) + O(N^{-1}), a \neq b \tag{18}$$

$$Cov\{d_a^{(lN)}(\lambda_1), d_a^{(lN)}(-\lambda_2)\}$$

$$= pqC_{aa}(0)R_{aa}\Phi_{aa}^{(lN)}(\lambda_1+\lambda_2) \tag{19}$$

$$+ p^2 R_{aa}\Phi_{ab}^{(lN)}(\lambda_1+\lambda_2) f_{aa}(\lambda_1) + O(N^{-1}),$$

at $\lambda_1 = \lambda_2 = \lambda$

$$Cov\{d_a^{(lN)}(\lambda), d_b^{(lN)}(-\lambda)\} \tag{20}$$

$$= \ p^2 R_{ab}\Phi_{ab}^{(lN)}(0) f_{ab}(\lambda_1) + O(N^{-1})$$

where $O(N^{-1})$ is uniform in λ as $N \to \infty$.

$$Cum\{d_{a_1}^{(lN)}(\lambda_1), ..., d_{a_k}^{(lN)}(\lambda_k)\}$$

$$= (2\pi)^{\frac{k}{2}-1} p^k R_{a_1...a_k}\Phi_{a_1...a_k}^{(lN)}(\sum_{j=1}^{k}\lambda_j) f_{a_1 a_2 ... a_k}(\lambda_1, ..., \lambda_k) + O(N^{-\frac{k}{2}}) \tag{21}$$

where $O(N^{-\frac{k}{2}})$ is uniform in $\lambda_1, \lambda_2, ..., \lambda_{k-1}$ as $N \to \infty$, $k = 2, ...$ and

$$R_{a_1...a_k} = \left(\sum_{t_1=lN}^{(l+1)N-1} \Phi_{a_1}^2(t_1 - lN)\right)^{-\frac{1}{2}} ... \left(\sum_{t_2=lN}^{(l+1)N-1} \Phi_{a_k}^2(t_1 - lN)\right)^{-\frac{1}{2}}$$

Proof. The proof of (17) comes directly from (13), since $E\{Y(t)\} = 0$.

$$Cov\{d_a^{(lN)}(\lambda_1), d_b^{(lN)}(-\lambda_2)\}$$

$$= E\{d_a^{(lN)}(\lambda_1) d_b^{(lN)}(\lambda_2)\}$$

$$= (2\pi)^{-1} R_{ab} E\{\sum_{t_1=lN}^{(l+1)N-1} \Phi_a(t_1 - lN)\exp(-i\lambda_1 t_1) Y_a(t_1)$$

$$\times \sum_{t_2=lN}^{(l+1)N-1} \Phi_b(t_2 - lN)\exp(-i\lambda_2 t_2) Y_b(t_2)\}$$

$$= (2\pi)^{-1} R_{ab}[\sum_{t=lN}^{(l+1)N-1} \Phi_a(t - lN)\Phi_b(t - lN)\exp(-i(\lambda_1+\lambda_2)t) E\{Y_a(t) Y_b(t)\}$$

$$+\sum_{t_1=lN}^{(l+1)N-1} \neq \sum_{t_2=lN}^{(l+1)N-1} \Phi_a(t_1-lN)\Phi_b(t_2-lN)\exp(-i(\lambda_1 t_1+\lambda_2 t_2)E\{Y_a(t_1)Y_b(t_2)\}]$$

let $a=b$, then

$$Cov\{d_a^{(lN)}(\lambda_1), d_a^{(lN)}(-\lambda_2)\}$$

$$=(2\pi)^{-1}pR_{aa}C_{aa}(0)\Phi_{aa}^{(lN)}(\lambda_1+\lambda_2)$$

$$+(2\pi)^{-1}p^2R_{aa}\sum_{t_1=lN}^{(l+1)N-1} \neq \sum_{t_2=lN}^{(l+1)N-1}[\Phi_a(t_1-lN)\Phi_b(t_2-lN)$$

$$\exp(-i(\lambda_1 t_1+\lambda_2 t_2)C_{aa}(t_1-t_2)].$$

let $t_1-t_2=u$, $u\neq 0$, $t_2=t$, and by Assumption II and (14), then

$$Cov\{d_a^{(lN)}(\lambda_1), d_a^{(lN)}(-\lambda_2)\}$$

$$=(2\pi)^{-1}pR_{aa}C_{aa}(0)\Phi_{aa}^{(lN)}(\lambda_1+\lambda_2)$$

$$+(2\pi)^{-1}p^2R_{aa}\Phi_{aa}^{(lN)}(\lambda_1+\lambda_2)\sum_{0\neq u=-(N+1)}^{(l+1)N-1}\exp(-i\lambda_1 u)C_{aa}(u)+O(1)$$

by (3) the proof of (19) is completed. We can get the proof of (18) and (20)by the same structure in (19). Since

$$Cum\{d_{a_1}^{(lN)}(\lambda_1),...,d_{a_k}^{(lN)}(\lambda_k)\}$$

$$=Cum\{\left(2\pi\sum_{t_1=lN}^{(l+1)N-1}\Phi_{a_1}^2(t_1-lN)\right)^{-\frac{1}{2}}\times$$

$$\sum_{t=lN}^{(l+1)N-1}\Phi_{a_1}(t_1-lN)\exp(-i\lambda t_1)Y_{a_1}(t_1),...,$$

$$\left(2\pi\sum_{t_k=lN}^{(l+1)N-1}\Phi_{a_1}^2(t_k-lN)\right)^{-\frac{1}{2}}\times\sum_{t_k=lN}^{(l+1)N-1}\Phi_{a_k}(t_k-lN)\exp(-i\lambda t_k)Y_{a_k}(t_k)\}$$

$$=(2\pi)^{-\frac{k}{2}}\left(\sum_{t_1=lN}^{(l+1)N-1}\Phi_{a_1}^2(t_1-lN)\right)^{-\frac{1}{2}}...\left(\sum_{t_k=lN}^{(l+1)N-1}\Phi_{a_1}^2(t_k-lN)\right)^{-\frac{1}{2}}$$

$$\times\sum_{t_1=lN}^{(l+1)N-1}...\sum_{t_k=lN}^{(l+1)N-1}\Phi_{a_1}(t_1-lN)...\Phi_{a_k}(t_k-lN)$$

$$\exp(-i\sum_{j=1}^{k}\lambda_j t_j)Cum\{Y_{a_1}(t_1),...,Y_{a_k}(t_k)\}$$

$$= (2\pi)^{-\frac{k}{2}} R_{a_1 \dots a_k} \sum_{t_1=lN}^{(l+1)N-1} \dots \sum_{t_k=lN}^{(l+1)N-1} \Phi_{a_1}(t_1 - lN) \dots \Phi_{a_k}(t_k - lN) \exp(-i \sum_{j=1}^{k} \lambda_j t_j)$$

$\times C_{a_1 \dots a_k}(t_1 - t_k, \dots, t_{k-1} - t_k)$

let $t_j - t_k = u_j$, $t_k = t$, $j = 1, \dots, k-1$ and by Lemma (P.4.1) in Brillinger (2001) PP. 420 and (6), we get the proof of (21). □

Corollary 4.1. Under the conditions of Theorem 4.1, we have

$$E\{d_a^{(lN)}(\pm\lambda_j)\} = 0,$$

$$N^{-1} Cov\{d_a^{(lN)}(\pm\lambda_j), d_b^{(lN)}(\pm\lambda_l)\}$$

$$= N^{-1} P^2 R_{ab} \Phi_{ab}^{(lN)}(\pm\lambda_j \mp \lambda_l) f_{ab}(\pm\lambda_j) + O(N^{-2}),$$

that tends to 0 as $N \to \infty$ if $\lambda_j \pm \lambda_l \not\equiv 0$ (mod2π), it tends to $2\pi P^2 R_{ab} \Phi_{ab}^{(T)}(0) f_{ab}(\pm\lambda_j)$ if $\pm\lambda_j \equiv \pm\lambda_l(\text{mod}2\pi)$.

$$N^{-\frac{k}{2}} Cum\{d_{a_1}^{(lN)}(\pm\lambda_{j_1}), \dots, d_{a_k}^{(lN)}(\pm\lambda_{j_k})\}$$

$$= N^{-\frac{k}{2}} (2\pi)^{\frac{k}{2}-1} p^k R_{a_1 \dots a_k} \Phi_{a_1 \dots \dots a_k}^{(lN)}(\pm\lambda_{j_1}(T) \pm \dots$$

$$\dots \pm \lambda_{j_k}(T)) f_{a_1 a_2 \dots a_k}(\pm\lambda_{j_1}(T), \dots, \pm\lambda_{j_k}(T)) + O(N^{-k}),$$

that tends to 0 as $T \to \infty$ if $k > 2$.

PROOF. The proof comes directly by Theorem 4.1. □

Theorem 4.2. Let $X(t)(t = 0, \pm 1, \dots .)$ be a zero mean strictly stationary r-vector valued time series satisfy Assumption I and $d_Y^{(T)}(\lambda_j)$ be given by (13). Suppose $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0$ (mod2π) for $1 \leq j < k \leq J$. Then $d_Y^{(lN)}(\lambda_j)$, $\lambda_j \not\equiv 0$ (mod2π), $j = 1, 2, \dots J$ are asymptotically independent $N_r^c(0, 2\pi p^2 R_{ab} \Phi_{ab}^{(lN)}(0) f_{ab}(\lambda_j))$, variates. Also if $\lambda = 0, \pm 2\pi, \dots$, $d_Y^{(T)}(\lambda)$ is asymptotically $N_r(0, 2\pi p^2 R_{ab} \Phi_{ab}^{(lN)}(0) f_{ab}(\lambda))$ independent of previous variates and if $\lambda = \pm\pi, \pm 3\pi \dots$, is asymptotically $N_r(0, 2\pi p^2 R_{ab}$-$\Phi_{ab}^{(lN)}(0) f_{ab}(\lambda))$ independent of previous variates.

PROOF. The proof comes directly by Corollary 4.1. □

5. THE SMOOTHED SPECTRAL DENSITY ESTIMATE

Using finite Fourier transform (13), we construct the modified periodogram as

$$I_{YY}^{(lN)}(\lambda) = (P^2 R_{YY} \Phi_{YY}^{(lN)}(0))^{-1} d_Y^{(T)}(\lambda) \overline{d_Y^{\prime(T)}(\lambda)}, \tag{22}$$

where the bar denotes the complex conjugate. The smoothed spectral density estimate is constructed as

$$f_{ab}^{(T)}(\lambda) = \frac{1}{L} \sum_{l=0}^{L-1} I_{ab}^{(lN)}(\lambda), a, b = 1, 2, \dots, r. \tag{23}$$

Theorem 5.1. Let $X(t)(t=0,\pm1,....)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{(T)}(\lambda)=\{I_{ab}^{(T)}(\lambda)\}_{a,b=1,2,...,r}$ be given by (22), and $\Phi_a(t)$ satisfy Assumption II for $a=1,2,...,r$, then

$$E\{I_{ab}^{(lN)}(\lambda)\}=f_{ab}(\lambda)+O(N^{-1}),p\rightarrow 1 \quad (24)$$

$$Cov\{I_{a_1b_1}^{(lN)}(\lambda_1),I_{a_2b_2}^{(lN)}(\lambda_2)\}$$

$$=(R_{a_1b_1}R_{a_2b_2}\Phi_{a_1b_1}^{(lN)}(0)\Phi_{a_2b_2}^{(lN)}(0))^{-1}$$

$$\times[R_{a_1a_2}R_{b_1b_2}\Phi_{a_1a_2}^{(lN)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{(lN)}(\lambda_1-\lambda_2)}f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \quad (25)$$

$$+R_{a_1b_2}R_{b_1a_2}\Phi_{a_1b_2}^{(lN)}(\lambda_1+\lambda_2)\overline{\Phi_{b_1a_2}^{(lN)}(\lambda_1+\lambda_2)}f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1)$$

$$+(2\pi)R_{a_1b_1a_2b_2}\Phi_{a_1b_1a_2b_2}^{(lN)}(0)f_{a_1b_1a_2b_2}(\lambda_1,-\lambda_1,\lambda_2)]+O(N^{-1})$$

$$Cum\{I_{a_1b_1}^{(lN)}(\lambda_1),...,I_{a_kb_k}^{(lN)}(\lambda_k)\}$$

$$=(\prod_{i=1}^{k}R_{a_ib_i}\Phi_{a_ib_i}^{(lN)}(0))^{-1}\sum\{\prod_{j=1}^{k}R_{a_jb_j}\Phi_{c_jd_j}^{(lN)}(\mu_j+\gamma_j)\}\{\prod_{j=1}^{k}f_{c_jd_j}(\mu_j)\} \quad (26)$$

$$+O(N^{-1})$$

where the summation extends over all partitions

$$\{(c_1,\mu_1),(d_1,\gamma_1)\},...,\{(c_k,\mu_k),(d_k,\gamma_k)\},$$

into pairs of the quantities $(a_1,\lambda_1),(b_1,-\lambda_1),....,(a_k,\lambda_k),(b_k,-\lambda_k)$ excluding the case with $\mu_j=-\gamma_j=\lambda_m$ for some j,m,where $O(N^{-1})$ is uniform in $\lambda_1,...,\lambda_k$.

Proof. By (22), we have

$$E\{I_{ab}^{(lN)}(\lambda)\}=(p^2R_{ab}\Phi_{ab}^{(lN)}(0))^{-1}E\{d_a^{(lN)}(\lambda)\overline{d_b^{'(lN)}(\lambda)\}}$$

$$=Cov\{d_a^{(lN)}(\lambda),d_b^{(lN)}(\lambda)\}$$

then by (18) the proof of (24) is completed. From (22), and by Theorem 2.3.2 in Brillinger (2001) PP.21, we have

$$Cov\{I_{a_1b_1}^{(lN)}(\lambda_1),I_{a_2b_2}^{(lN)}(\lambda_2)\}$$

$$=Cov\{d_{a_1}^{(lN)}(\lambda_1)d_{b_1}^{'(lN)}(-\lambda_1),d_{a_2}^{(lN)}(\lambda_2)d_{b_2}^{'(lN)}(-\lambda_2)\}$$

$$=Cum\{d_{a_1}^{(lN)}(\lambda_1),d_{b_1}^{(lN)}(-\lambda_1),d_{a_2}^{(lN)}(\lambda_2),d_{b_2}^{(lN)}(-\lambda_2)\}$$

$$+Cov\{d_{a_1}^{(lN)}(\lambda_1),d_{a_2}^{(lN)}(\lambda_2)\}Cov\{d_{b_1}^{(lN)}(-\lambda_1),d_{b_2}^{(lN)}(-\lambda_2)\}$$

$$+Cov\{d_{a_1}^{(lN)}(\lambda_1), d_{b_2}^{(lN)}(-\lambda_2)\}Cov\{d_{b_1}^{(lN)}(-\lambda_1), d_{a_2}^{(lN)}(\lambda_2)\}.$$

By Theorem 4.1 the proof of (25) is completed. From (22), we have

$$Cum\{I_{a_1b_1}^{(lN)}(\lambda_1), ..., I_{a_kb_k}^{(lN)}(\lambda_k)\}$$

$$= (\prod_{i=1}^{k} R_{a_ib_i}\Phi_{a_ib_i}^{(lN)}(0))^{-1}Cum\{d_{a_1}^{(lN)}(\lambda_1)d_{b_1}^{'(lN)}(-\lambda_1), ..., d_{a_k}^{(lN)}(\lambda_k)d_{b_k}^{'(lN)}(-\lambda_k)\}$$

By Theorem 2.3.2 in Brillinger (2001) PP.21, we get

$$Cum\{d_{a_1}^{(lN)}(\lambda_1)d_{b_1}^{'(lN)}(-\lambda_1), ..., d_{a_k}^{(lN)}(\lambda_k)d_{b_k}^{'(lN)}(-\lambda_k)\}$$

$$= \sum_{\nu} Cum\{d^{(lN)}(\lambda_i); i \in \nu_1\}...Cum\{d^{(lN)}(\lambda_i); i \in \nu_s\},$$

where the summation extends over all indecomposable partitions $\nu = [\cup_{j=1}^{s}\nu_j] \in I, I = (a_1, ..., a_k\ ;\ b_1, ...b_k)$, $1 \leq s \leq k$ of the transformed table

$$\begin{array}{llcll} (a_1, \lambda_1), & (b_1, -\lambda_1) & & \{(c_1, \mu_1), & (d_1, \gamma_1)\} \\ (a_2, \lambda_2), & (b_2, -\lambda_2) & & \{(c_2, \mu_2), & (d_2, \gamma_2)\} \\ . & . & \rightarrow & . & . \\ . & . & & . & . \\ (a_k, \lambda_k), & (b_k, -\lambda_k) & & \{(c_k, \mu_k), & (d_k, \gamma_k)\}. \end{array}$$

Then, by Theorem (4.1), we get

$$Cum\{I_{a_1b_1}^{(lN)}(\lambda_1), ..., I_{a_kb_k}^{(lN)}(\lambda_k)\}$$

$$= (\prod_{i=1}^{k} R_{a_ib_i}\Phi_{a_ib_i}^{(lN)}(0))^{-1}\sum_{\nu}\Pi_{j=1}^{k}\{R_{a_jb_j}(\Phi_{c_jd_j}^{(T)}(\mu_j+\gamma_j)f_{c_jd_j}(\mu_j) + O(N^{-1})\}.$$

By Lemma 4.1, the proof of (26) is completed. □

Theorem 5.2. Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{(lN)}(\lambda) = \{I_{ab}^{(lN)}(\lambda)\}_{a,b=1,2,...,r}$ be given by (22), $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0 \pmod{2\pi}$ for $1 \leq j < k \leq J$ and $\Phi_a(t)$ satisfy Assumption II for $a = 1, 2, ..., r,$. Then $I_{YY}^{(lN)}(\lambda_j)$, $j = 1, 2, ..J$ are asymptotically independent $W_r^c(1, f_{XX}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi...$ then $I_{YY}^{(lN)}(\lambda)$ is asymptotically $W_r(1, f_{XX}(\lambda))$ independent of the previous variates.

Proof. The proof comes directly from Theorem 4.2, for more details about Wishart distribution see Anderson (1971). □

Theorem 5.3. Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{(lN)}(\lambda)$ be given by (23), $a, b = 1, 2, ..., r$, then

$$E\{f_{ab}^{(T)}(\lambda)\} = f_{ab}(\lambda) + O(N^{-1}) \tag{27}$$

$$Cov\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\} = O(1) \tag{28}$$

PROOF. By (23), we have

$$E\{f_{ab}^{(T)}(\lambda)\} = \frac{1}{L}\sum_{l=0}^{L-1} E\{I_{ab}^{(lN)}(\lambda)\}$$

then by (24) the proof of (27) is completed. From (23), we get

$$Cov\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\}$$

$$= \frac{1}{L^2}\sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1} Cov\{I_{a_1b_1}^{(lN)}(\lambda_1), I_{a_2b_2}^{(lN)}(\lambda_2)\}.$$

Since

$$\left|Cov\{I_{a_1b_1}^{(lN)}(\lambda_1), I_{a_2b_2}^{(lN)}(\lambda_2)\}\right|$$

$$\leq M\{|R_{a_1a_2}R_{b_1b_2}|\left|\Phi_{a_1a_2}^{(lN)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{(lN)}(\lambda_1-\lambda_2)}\right||f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1)|$$

$$+|R_{a_1b_2}R_{b_1a_2}|\left|\Phi_{a_1b_2}^{(lN)}(\lambda_1+\lambda_2)\overline{\Phi_{b_1a_2}^{(lN)}(\lambda_1+\lambda_2)}\right||f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1)|$$

$$+\left|R_{a_1b_1a_2b_2}\Phi_{a_1b_1a_2b_2}^{(lN)}(0)f_{a_1b_1a_2b_2}(\lambda_1,-\lambda_1,\lambda_2)\right|\} + O(N^{-1}),$$

and by Lemma 4.1, the proof of (28) is completed. □

Theorem 5.4. Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{(lN)}(\lambda)$ be given by (23), $a, b = 1, 2, ..., r$, $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0 \pmod{2\pi}$ for $1 \leq j < k \leq J$, Then $Lf_{ab}^{(lN)}(\lambda_j)$, $j = 1, 2, ..J$ are asymptotically independent $W_r^c(L, f_{ab}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi...$ then $Lf_{ab}^{(lN)}(\lambda)$ is asymptotically $W_r(L, f_{ab}(\lambda))$ independent of the previous variates.

PROOF. The proof comes directly by Theorem (5.2) and Theorem 7.3.2 in Anderson (1971) PP.162. □

6. EXAMPLE

One of the most important reasons of the current wide-spread of spectral analysis is the increasing availability of powerful electronic computers, which allow practitioners to carry out the necessary computations. In this section, we use the data that was introduced by Jones et al (1999) to discuss our results and to compare between our results, spectral analysis of strictly stationary time series with some missing observations, and the classical results, where all observations are available. Data is available at *http//www.cru.uea.ac.uk*. Using Dirichlet data window $\Phi_a(t) = 1$, and for $\lambda = \frac{2\pi k}{N}, k = 0, 1, ..., \frac{N}{2}$, the finite Fourier transform will be in the form

$$d_Y^{(T)}(\lambda) = (2\pi N)^{-\frac{1}{2}} \times \sum_{t=(l-1)N}^{lN-1} \exp\left(\frac{-2\pi ikt}{N}\right)Y(t), l = 1, ..., L.$$

For $L = 10$ and $N = 12$ the spectral density estimate is computed in two cases, where all observations are available i.e. $p = 1$ and where there are some randomly missing values with $p = 0.95$ and $p = 0.98$.

Via simulation we can see that the spectral density function in the case where there are some randomly missing values approaches to the one where all observations are available as $p \to 1$.

Let $X(t) = (X_1(t), X_2(t))'$, where $X_1(t)$ is the Northern Hemisphere Monthly Temperature Anomalies (degrees C) (1990-1999) and $X_2(t)$ is the Southern Hemisphere Monthly Temperature Anomalies (degrees C) (1990-1999). First, we consider the case where all observations are available i.e. $p = 1$, $Y(t) = X(t)$, which is the classic case and then we suppose that there is some missing observations in randomly way, i.e. $p \neq 1$, to compare two cases. The computations are made for different value of p. The value of p is estimated as $\frac{n_1+n_2}{2LN}$, where n_1, n_2 are the numbers of the observed data of $X_1(t), X_2(t)$ respectively, $p \to 1$ as $n_1, n_2 \to T$. ∇_{12} is used to smooth data before calculations, see Chatfield(1989)pp.18. Mathematica 4 package is used for calculations.

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Fig. 1

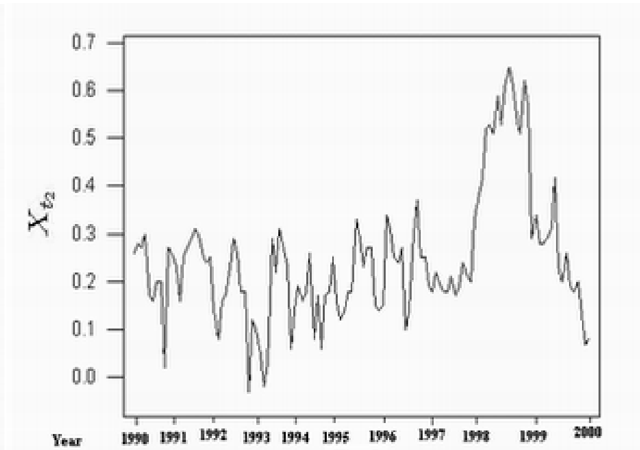

Fig. 2

• Fig. 1. and Fig. 2. show The Northern and Southern Hemisphere Monthly Temperature Anomalies (degrees C),(X_{t_1}, X_{t_2}), 1990-1999.

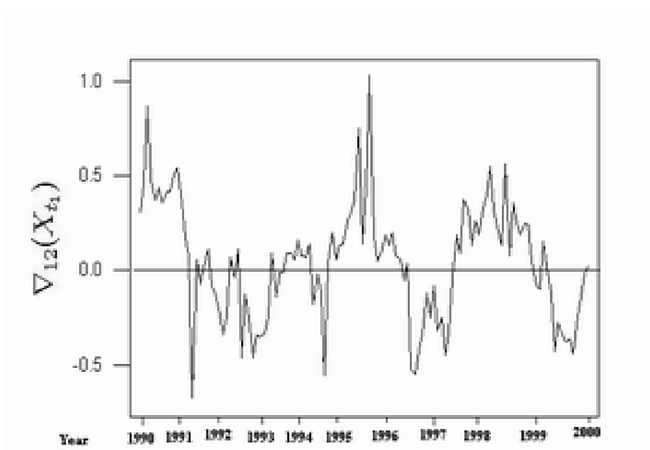

Fig.3

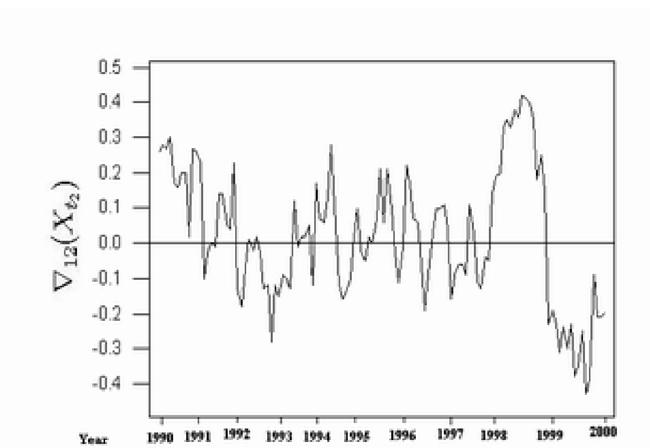

Fig.4

• Fig. 3. and Fig. 4., show Northern and Southern Hemisphere Monthly Temperature after adjustment $\nabla_{12}(X_{t_1})$,$\nabla_{12}(X_{t_2})$ respectively.

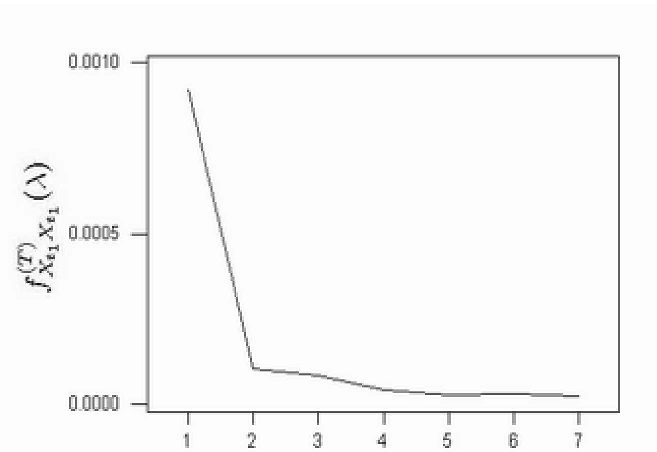

Fig. 5

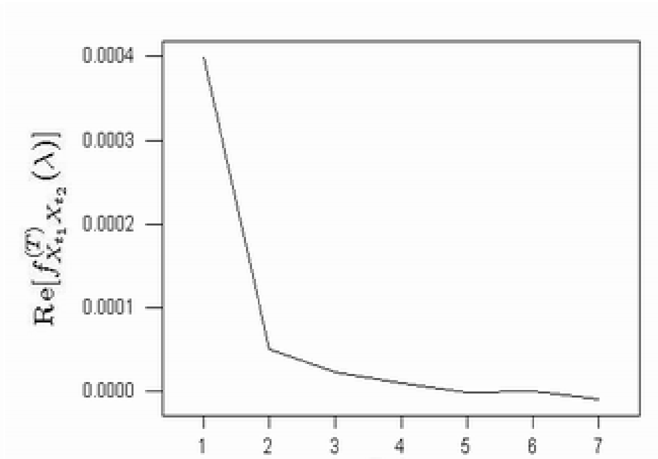

Fig. 6

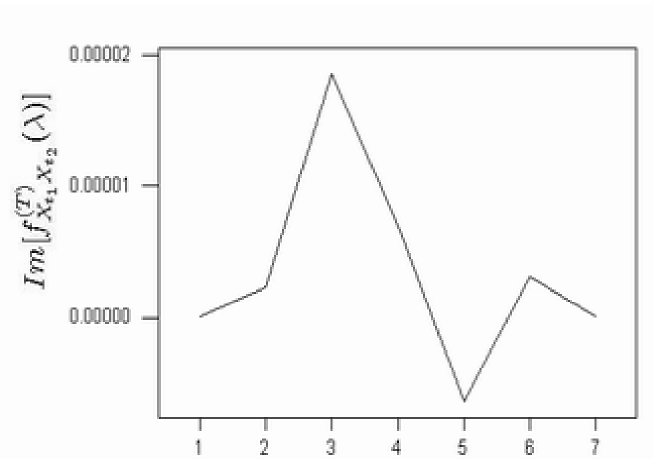

Fig. 7

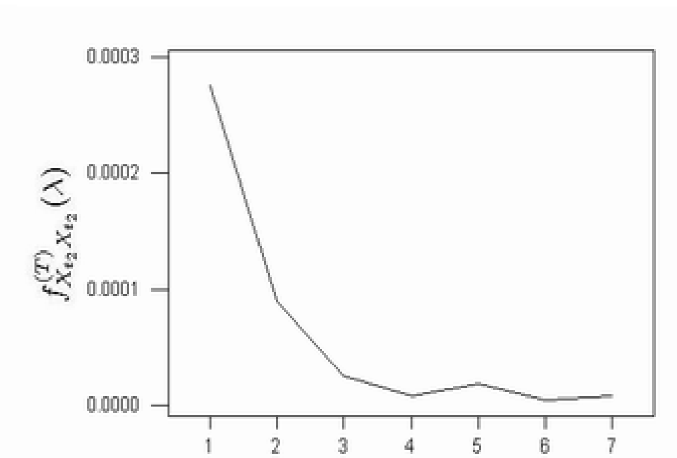

Fig. 8

• Fig. 5, Fig. 6, Fig. 7 and Fig. 8, show the spectral density function of $X_t = (X_{t1}, X_{t2})'$, for $\lambda = \frac{2\pi k}{N}, k = 0, 1, ..., \frac{N}{2}$ against k.

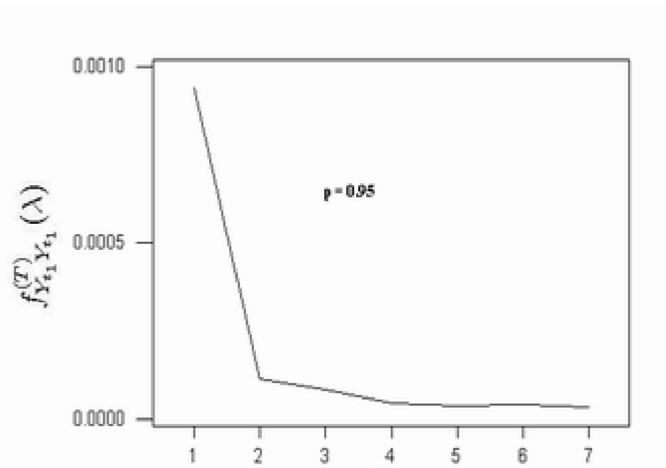

Fig.9

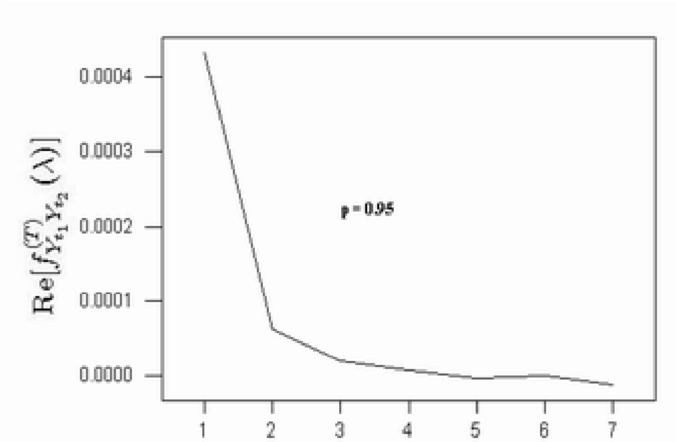

Fig.10

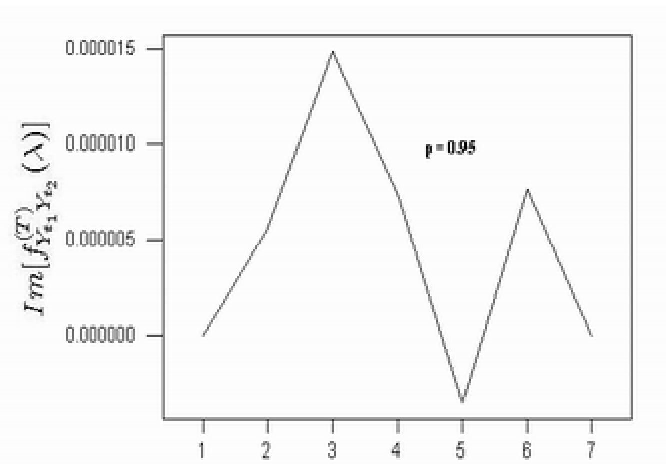

Fig.11

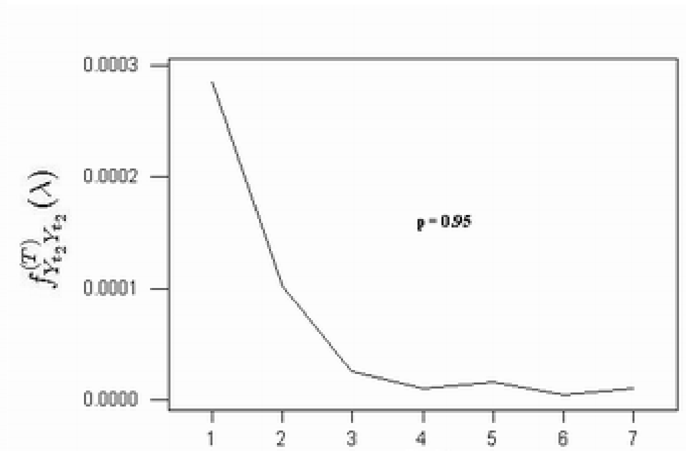

Fig.12

• Fig. 9, Fig.10, Fig. 11 and Fig. 12, show the spectral density function of $Y_t = (Y_{t1}, Y_{t2})'$, with $p = 0.95$.

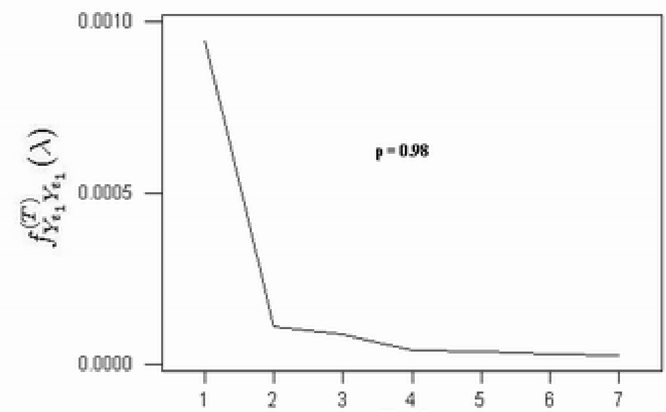

Fig. 13

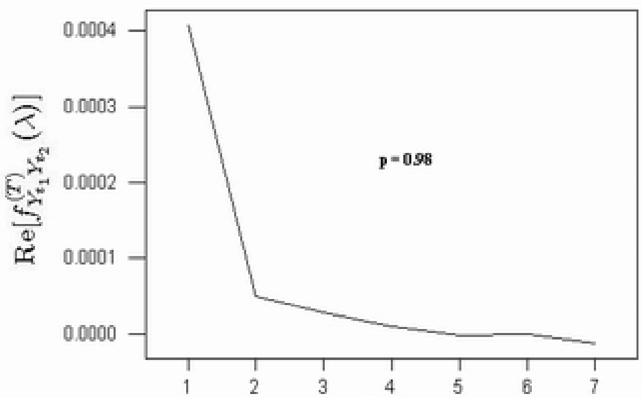

Fig. 14

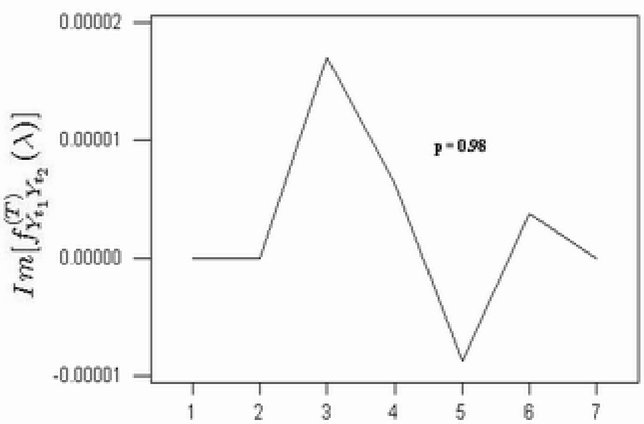

Fig. 15

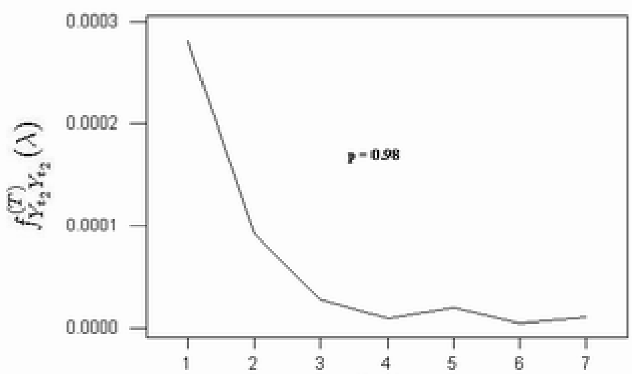

Fig. 16

• Fig. 13, Fig. 14, Fig. 15 and Fig. 16, show the spectral density function of $Y_t = (Y_{t1}, Y_{t2})'$, with $p = 0.98$.