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GENERALIZED R-NORM FUZZY INFORMATION MEASURE

D.S.HOODA AND DIVYA JAIN

Abstract

In the present paper, a generalized measure of R-norm fuzzy information is defined and its validity is proved. Particular cases of this new measure has been discussed. The monotonic behavior of parametric generalized measure with regard to its parameter is studied with tables and figures.

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Additional Key Words and Phrases: Fuzziness, ambiguity, R-norm information, monotonic behavior, fuzzy information.

1. INTRODUCTION

Uncertainty and fuzziness are the basic nature of human thinking and of many real world problems. Shannon defined "entropy" as a measure to estimate uncertain degree of the randomness. The concept of entropy has been widely used in different areas, e.g., communication theory, statistical mechanics, finance, pattern recognition, neural network and image processing etc. It has also been applied to other areas such as clustering, fuzzy logic systems, decision making and regression analysis.

Zadeh (1965), developed the concept of fuzzy set and defined the entropy of a fuzzy event, measure by considering the distance from a set to its nearest non-fuzzy set and the distance from the set to its farthest nonfuzzy set. The concept of entropy was developed to measure the uncertainty of a probability distribution. Zadeh (1965) introduced the concept of fuzzy sets and developed the theory to measure the ambiguity of a fuzzy set A. Let $X=\left[x_1,x_2......x_n\right]$ be a set of universe of discourse. Fuzzy set *A* is represented as:

$$A=\{x_i \mid \mu_A(x_i)\in[0,1]: x_i\in X \ \ \forall i=1,2,...,n\},$$

where $\mu_A(x_i)$ is a membership function defined as follows:

$$\mu_A(x_i)=\begin{cases}0, \ if \ x_i \ \text{ does not belong to } A \text{ and there is no ambiguity,}\\ 1, \ if \ x_i \ \text{ belongs to } A \ \text{ and there is no ambiguity,}\\ 0.5, if \ \text{ there is maximum ambiguity whether } x_i \text{ belongs to } A \text{ or not.}\end{cases}$$

In fact $\mu_A(x_i)$ associates with each $x_i\in X$ a grade of membership to the set *A*.

Fuzzy entropy is the measurement of fuzziness in a fuzzy set. It has wide applications in the area of pattern recognition, image processing, speech recognition, etc. It is often required to get some idea about the degree of fuzziness or ambiguity present in a fuzzy set. So, it has especial position in fuzzy systems, such as fuzzy pattern recognition systems, fuzzy neural network systems, fuzzy image processing fuzzy knowledge base systems, fuzzy decision making systems, fuzzy control systems and fuzzy management information systems.

If $x_1, x_2, ..., x_n$ are members of the universe of discourse, then all $\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$ lie between 0 and 1, but these are not probabilities because their sum is not unity. However,

$$\Phi_A(x_i) = \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}, \ i = 1, 2, ..., n, \tag{1.1}$$

is a probability distribution. Thus Kaufman (1980) defined entropy of a fuzzy set *A* having *n* support points by:

$$H(A) = -\frac{1}{\log n} \sum_{i=1}^{n} \Phi_A(x_i) \log \Phi_A(x_i). \tag{1.2}$$

In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity or difficulty in making a decision whether an element belongs to a set or not. A measure of fuzziness $H(A)$ in a fuzzy set should have at least the following properties:

(P-1) $H(A)$ is minimum if and only if *A* is a crisp set, i.e. $\mu_A(x_i) = 0$ or 1 for all x_i.

(P-2) $H(A)$ is maximum if and only if *A* is most fuzzy set, i.e. $\mu_A(x_i) = 0.5$ for all x_i.

(P-3) $H(A) \geq H(A^*)$, where A^* is sharpened version of *A*.

(P-4) $H(A) = H(\overline{A})$, where $\overline{A}$ is the complement of *A*.

Since $\mu_A(x_i)$ and $1 - \mu_A(x_i)$ give the same degree of fuzziness, therefore corresponding to entropy due to Shannon (1948), Deluca (1971) suggested the following measure of fuzzy entropy:

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \log\left(1 - \mu_A(x_i)\right)\right]. \tag{1.3}$$

As (1.3) satisfies all four properties (P-1) to (P-4), so it is a valid measure of fuzzy entropy.

GENERALIZED R-NORM FUZZY INFORMATION MEASURE

In literature, a number of measures of fuzzy entropy corresponding to the various information measures have been proposed in order to combine the fuzzy set theory and its application to the entropy concept as fuzzy information measures.

Boekee & Lubbe (1980) proposed the following R-norm information measure:

$$H_R(P)=\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{\infty} p_i^{\,R}\right)^{\frac{1}{R}}\right] \quad ; \quad R>0, R\neq 1. \tag{1.4}$$

The measure (1.4) is called R-norm information measure and the most important property of this measure is that when $R \to 1$, R-norm information measure approaches to Shannon's entropy and in case $R \to \infty$, $H_R(P) \to (1-\max p_i); i=1,2......n.$

Corresponding to measure (1.4) Hooda (2004) proposed the following R-norm fuzzy information measure:

$$H_R(A)=\frac{R}{R-1}\sum_{i=1}^{\infty}\left[1-\left(\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right] \quad ; \quad R>0, R\neq 1. \tag{1.5}$$

Hooda and Ram (1998) characterized a generalized R-norm information measure of degree β as given below:

$$H_R^{\beta}(P)=\frac{R}{R+\beta-2}\left[1-\left(\sum_{i=1}^{\infty} p_i^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right] \quad ;$$

$$0<\beta<1, \quad R>0 \;\; and R+\beta\neq 2. \tag{1.6}$$

Corresponding to measure (1.6), Hooda and Bajaj (2008) defined a generalized measure of R-norm fuzzy information measure of degree β as given by

$$H_R^{\beta}(A)=\frac{R}{R+\beta-2}\sum_{i=1}^{\infty}\left[1-\left(\mu_A^{\frac{R}{2-\beta}}(x_i)+(1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right], \tag{1.7}$$

where $0<\beta<1, \;\; R(>0)\neq 1, \;\; R+\beta\neq 2.$

Further Hooda and Sharma (2008) generalized measure (1.6) and characterized the following R-norm entropy of order α and degree β as given below:

$$H_R^{(\alpha,\beta)}(P)=\frac{R}{R+\beta-2\alpha}\left[1-\left(\sum_{i=1}^{\infty} p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}\right], \tag{1.8}$$

where $\alpha \geq 1,\ 0<\beta \leq 1,\ \ R(>0)\neq 1,\ R+\beta \neq 2\alpha$.

In this paper we define a generalized measure of R-norm fuzzy information and prove its validity in section 2. In section 3 we study its monotonic behavior with regard to parameters.

2. GENERALIZED MEASURE OF R-NORM FUZZY INFORMATION

Corresponding to measure (1.8) characterized by Hooda and Sharma (2008), we propose the following R-norm fuzzy information measure:

$$H_R^{\alpha,\beta}(A)=\frac{R}{R-2\alpha+\beta}\sum_{i=1}^{n}\left[1-\left\{\mu_A^{\frac{R}{2\alpha-\beta}}(x_i)+(1-\mu_A(x_i))^{\frac{R}{2\alpha-\beta}}\right\}^{\frac{2\alpha-\beta}{R}}\right], \tag{2.1}$$

where $\alpha \geq 1,\ 0<\beta \leq 1,\ R(>0)\neq 1,\ \ R+\beta \neq 2\alpha$.

Theorem1. The generalized R-norm fuzzy information measure given by (2.1) is a valid measure of fuzzy information.

Proof: To prove that the measure (2.1) is a valid measure, we shall show that it is satisfying the four properties (P-1) to (P-4).

The measure (2.1) can be rewritten as:

$$H_R^{\alpha,\beta}(A)=\lambda\sum_{i=1}^{\infty}\left[1-\left\{\mu_A^{\rho}(x_i)+(1-\mu_A(x_i))^{\rho}\right\}^{\frac{1}{\rho}}\right], \tag{2.2}$$

where $\lambda=\dfrac{R}{R-2\alpha+\beta}, \rho=\dfrac{R}{2\alpha-\beta}(>0)\neq 1$.

(P-1) Sharpness

We know that $\mu_A^{\rho}(x_i)+(1-\mu_A(x_i))^{\rho}\le 1$, $\forall\,\mu_A(x_i)$ and the equality holds only if $\mu_A(x_i)=0\ or\ 1$. Hence $H_R^{\alpha,\beta}(A)\ge 0$ for all $\mu_A(x_i)$ and $H_R^{\alpha,\beta}(A)=0$ if and only if $\mu_A(x_i)=0\ or\ 1$ i.e. if and only if A is the most fuzzy set or crisp set.

(P-2) Maximality

Differentiating $H_R^{\alpha,\beta}(A)$ with respect to $\mu_A(x_i)$, we get

$$\frac{dH_R^{\alpha,\beta}(A)}{d(\mu_A(x_i))}=-\lambda\left[\sum_{i=1}^{\infty}\left\{\mu_A^{\rho}(x_i)+(1-\mu_A(x_i))^{\rho}\right\}\right]^{\frac{1-\rho}{\rho}}\left\{\mu_A^{\rho-1}(x_i)-(1-\mu_A(x_i))^{\rho-1}\right\} \tag{2 .3}$$

Let $0\le\mu_A(x_i)<0.5,$ then two cases arises.

Case1. $R>2\alpha-\beta$

In this case we have $\lambda>0,\ \ \rho>1$ and $\mu_A^{\rho-1}(x_i)-(1-\mu_A(x_i))^{\rho-1}<0$,

which implies $\dfrac{dH_R^{\alpha,\beta}(A)}{d(\mu_A(x_i))}>0$.

Case2. $R<2\alpha-\beta$

In this case we have $\lambda<0,\ \ \rho<1$ and $\mu_A^{\rho-1}{}_A(x_i)-(1-\mu_A(x_i))^{\rho-1}>0$,

which implies $\dfrac{dH_R^{\alpha,\beta}(A)}{d(\mu_A(x_i))}>0$.

Hence $H_R^{\alpha,\beta}(A)$ is a increasing function of $\mu_A(x_i)$ in the region $0\le\mu_A(x_i)<0.5$. Similarly, it can be proved that $H_R^{\alpha,\beta}(A)$ is a decreasing function of $\mu_A(x_i)$ in the region

$0.5<\mu_A(x_i)\le 1$. It is clear from (2.3) that it vanishes at $\mu_A(x_i)=0.5$. It shows that $H_R^{\alpha,\beta}(A)$ is a concave function and its maximum value exist at $\mu_A(x_i)=0.5$ i.e. maxima exist if and only if $\mu_A(x_i)$ is the most fuzzy set.

(P-3) Sharpening reduces the fuzzy information measure

Since $H_R^{\alpha,\beta}(A)$ is a increasing function of $\mu_A(x_i)$ in the region $0\le\mu_A(x_i)<0.5$ and is a decreasing function of $\mu_A(x_i)$ in the region $0.5<\mu_A(x_i)\le 1$, therefore,

$$\mu_{A^*}(x_i)\le\mu_A(x_i)\Rightarrow\ H_R^{\alpha,\beta}(A^*)\le H_R^{\alpha,\beta}(A)\ \text{in}\ 0\le\mu_A(x_i)<0.5 \tag{2.4}$$

$$\mu_{A^*}(x_i)\ge\mu_A(x_i)\Rightarrow\ H_R^{\alpha,\beta}(A^*)\le H_R^{\alpha,\beta}(A)\ \text{in}\ 0.5<\mu_A(x_i)\le 1 \tag{2.5}$$

From (2.4) and (2.5) we have $H_R^{\alpha,\beta}(A*) \le H_R^{\alpha,\beta}(A)$.

(P-4) Symmetry

Obviously from the definition of $H_R^{\alpha,\beta}(A)$, we have $H_R^{\alpha,\beta}(\overline{A}) = H_R^{\alpha,\beta}(A)$.

Since $H_R^{\alpha,\beta}(A)$ satisfies all the properties of fuzzy information measure, therefore it is a valid measure of fuzzy information. We can call (2.1) measure as the generalized R-norm fuzzy information measure of type α degree β.

Particular Cases:

(i) In case $\alpha = 1$, (2.1) reduces to (1.7).

(ii) In case $\alpha=1$ and $\beta = 1$, (2.1) reduces to (1.5).

(iii) In case $\alpha=1$, $\beta = 1$ and $R \to 1$, (2.1) reduces to (1.3).

(iv) In case $\alpha=1$, $\beta = 1$ and $R \to \infty$, (2.1) reduces to

$$\sum_{i=1}^{n}\left[1 - \max\{\mu_A(x_i),\ \ 1-\mu_A(x_i)\}\right]. \tag{2.6}$$

3. MONOTONIC BEHAVIOR OF R-NORM FUZZY INFORMATION MEASURE OF TYPE α DEGREE β

Let A_1 = (0.2,0.3, 0.4,0.3,0.2); A_2 = (0.2,0.3, 0.3,0.3,0.3) and A_3 = (0.2,0.3, 0.4,0.4,0.2) be any three standard fuzzy sets. Let us assume R as 0.6, α = 1.5 and different values of β. The graph (3.1) of $H_R^{\alpha,\beta}(A)$ with respect to β is plotted based on the following computed table (3.1):

Table 3.1

β	$H_{0.6}^{1.5,\beta}(A_1)$	$H_{0.6}^{1.5,\beta}(A_2)$	$H_{0.6}^{1.5,\beta}(A_3)$
0.1	8041.519	8042.519	8041.619
0.2	5733.022	5735.022	5733.122
0.3	4096.289	4098.289	4096.389
0.4	2932.95	2934.95	2933.95
0.5	2105.468	2104.468	2106.468
0.6	1515.646	1516.646	1516.646
0.7	1094.366	1093.366	1094.366
0.8	792.8693	793.8693	792.8693
0.9	576.5579	575.5579	577.5579

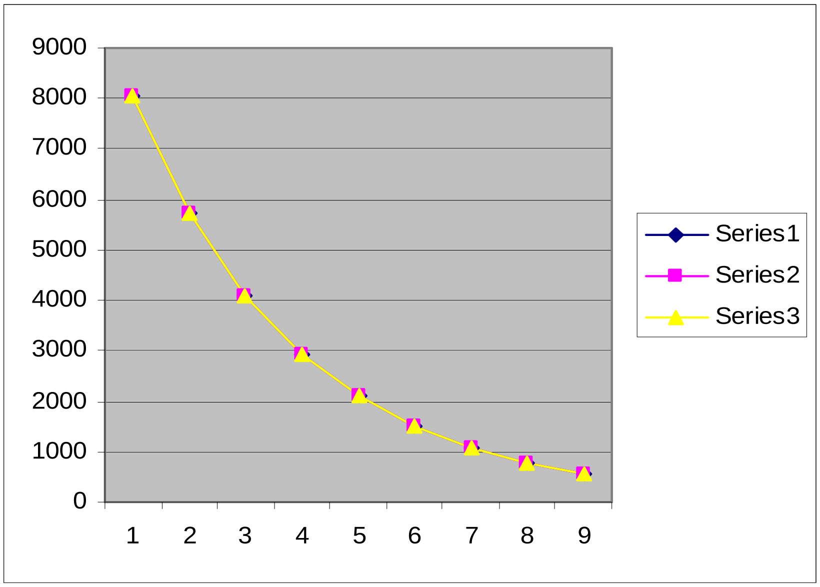

Fig. 3.1

From fig. 3.1, it is clear that the function is monotonic decreasing with respect to β for given value of *R=0.6 and* $\alpha = 1.5$.

Let A_1 = (0.2,0.3, 0.4,0.3,0.2); A_2 = (0.2,0.3, 0.3,0.3,0.3) and A_3 = (0.2,0.3, 0.4,0.4,0.2) be any three standard fuzzy sets. Let us assume R as 0.6, β = 1.5 and different values of α. The graph (3.1) of $H_R^{\alpha,\beta}(A)$ with respect to α is plotted based on the following computed table (3.2):

Table 3.2

α.	$H_{0.6}^{\alpha,0.1}(A_1)$	$H_{0.6}^{\alpha,0.1}(A_2)$	$H_{0.6}^{\alpha,0.1}(A_3)$
1.5	8041.519309	8042.519309	8041.719309
2	258322.9751	258323.9751	258322.9752
5	9.04963E+14	9.04964E+14	9.04983E+14
50	8.4144E+163	8.42E+163	8.42E+163

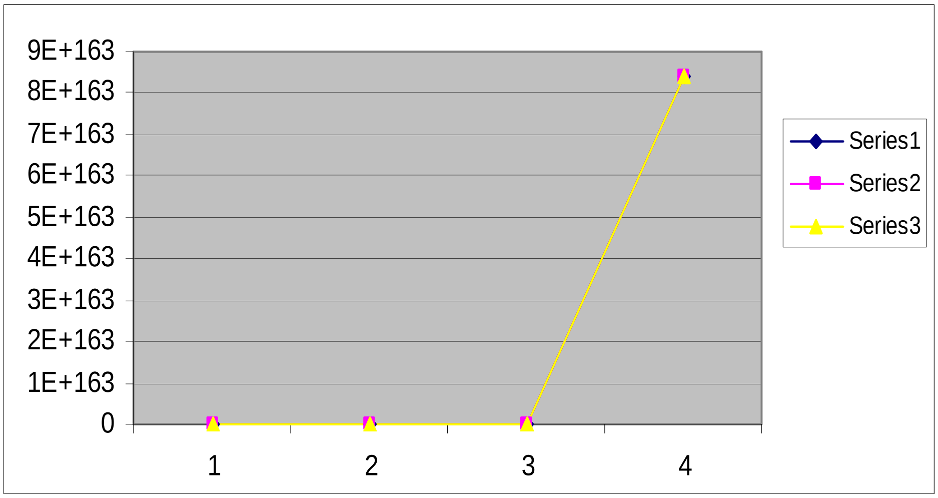

Fig:3.2

From fig. 3.3, it is clear that the function is monotonic increasing with respect to α for given value of β=0.1, $R = 0.6$.

Let A_1 = (0.2,0.3, 0.4,0.3,0.2); A_2 = (0.2,0.3, 0.3,0.3,0.3) and A_3 = (0.2,0.3, 0.4,0.4,0.2) be any three standard fuzzy sets. Let us assume α=100, β= 0.1 and different values of R. The graph (3.1) of $H_R^{\alpha,\beta}(A)$ with respect to R is plotted based on the following computed table (3.3):

Table:3.3

	$H_R^{100,0.1}(A_1)$	$H_R^{100,0.1}(A_2)$	$H_R^{100,0.1}(A_3)$
0.6	8.4144E+163	8.42E+163	8.43E+163
4	1.7416E+23	1.74E+23	1.74E+23
25	1503.791394	1504.79139	1505.7914
200	6.509811538	6.51981154	6.5209812
2000	4.388179397	4.3981794	4.4181794

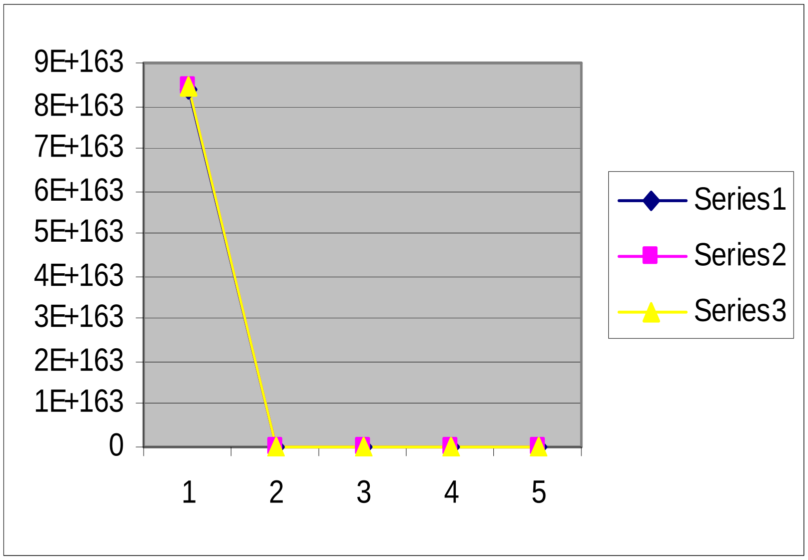

Fig. 3.3

From fig. 3.3, it is clear that the function is monotonic decreasing with respect to R for given value of β=0.1, $\alpha = 100$.

4. CONCLUSION

In present communication we have studied parametrically generalized R- norm fuzzy information measure which reduces to the known measure after particular values of parameters. Thus it is more flexible measure for application points of view. Its monotonic behavior gives very interesting results.

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D.S.Hooda
Jaypee University of Engineering and Technology
Guna
Email : ds_hooda@rediffmail.com

Divya Jain
Jaypee University of Engineering and Technology
Guna
Email : agrawal_divya2002@yahoo.co.in

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