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NEW FINITE DOUBLE INTEGRAL FORMULAE INVOLVING POLYNOMIALS AND FUNCTIONS OF GENERAL NATURE

PRAVEEN AGARWAL AND SHILPI JAIN

Abstract

The aim of the present paper is to establish two new finite double integral formulae involving product of the H-function of two variables**,** general sequence of function and the general class of polynomials**.** These double integral formulae are unified in nature and act as key formulae from which we can obtain as their special cases, double integral formulae concerning a large number of simpler special functions and polynomials. For the sake of illustration, we record here eight special cases of our main formulae which are also new and of interest by themselves. The findings of the present results are basic in nature and are likely to find useful applications in several fields.

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1. INTRODUCTION

The general sequence of function $R_n^{(\alpha,\beta)}$ **[x]** occurring in this paper is a special case of the sequence of functions given by Agarwal and Chaubey [1, p.1155] see also [11, p.447, problem (16)] and will be defined and represented in the following manner:

$$R_n^{(\alpha,\beta)}\left[x;a,b,c,d;p,q;\gamma,\delta;e^{-sx^r}\right]=\frac{b^{\gamma n}x^{kn}\left(cx^q+d\right)^{\delta n}k^n e^{sx^r}}{k'_n}\sum_{m,v,u,t,e}\frac{(-1)^{t+m}(-v)_u(-t)_e}{m!u!v!t!e!}$$
$$(\alpha)_t s^m\frac{(-\alpha-\gamma n)_e}{(1-\alpha-t)_e}(-\beta-\delta n)_v\left(\frac{pe+rm+\lambda+qu}{k}\right)_n\left(\frac{cx^q}{cx^q+d}\right)^v\left(\frac{ax^p}{b}\right)^v(x)^{rm} \tag{1.1}$$

where

$$\sum_{m,v,u,t,e}\equiv\sum_{m=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{e=0}^{t}, \tag{1.2}$$

and the infinite series on the right hand side of (1.1) is absolutely convergent.

The following explicit series form of $R_n^{(\alpha,\beta)}$ **[x]** for s=0 [3, p.544, Eq. (1.2)] will be required in the derivation of our main formulas:

$$R_n^{(\alpha,\beta)}\left[x;a,b,c,d;p,q;\gamma,\delta;1\right] = \sum_{v,u,t,e} \theta_1^*(v,u,t,e)\, x^{R'} \left(1+\frac{c}{d}x^q\right)^{-v+\delta n}, \tag{1.3}$$

where

$$\theta_1^*(v,u,t,e) = \frac{b^{\gamma n} k^n d^{\delta n - v}(-1)^t (-v)_u (-t)_e}{v!u!t!e!k'_n} (\alpha)_t (c)^v \left(\frac{a}{b}\right)^{pt} \frac{(-\alpha-\gamma n)_e}{(1-\alpha-t)_e} (-\beta-\delta n)_v \left(\frac{pe+\lambda+qu}{k}\right)_n . \tag{1.4}$$

$$R' = kn + qv + pt, \tag{1.5}$$

and

$$\sum_{v,u,t,e} \equiv \sum_{v=0}^{n} \sum_{u=0}^{v} \sum_{t=0}^{n} \sum_{e=0}^{t} . \tag{1.6}$$

It may be pointed out that $R_n^{(\alpha,\beta)}$ **[x]** unifies and extends a large number of named classical polynomials and polynomials studied by several research workers.
The general class of polynomials introduced by Srivastava [8, p.1, Eq. (1)]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k, \quad (n=0, 1, 2\ldots) \tag{1.7}$$

where m is arbitrary positive integers and the coefficients $A_{n,k}(n,k \geq 0)$ are arbitrary constants, real or complex. On suitable specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobin polynomials, the Laguerre polynomials, the Bessel's polynomials and several others [10, pp.158-161].

The H-function of several variables is defined and represented as follows [9, pp.251-252, Eqn's (C.1)-(C.3)]:

$$H\,[z_1,\ldots,z_r] \equiv H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q\,:\,p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p} : (c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ \\ (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q} : (d'_j,\delta'_j)_{1,q_1};\ldots;(f^{(r)}_j,F^{(r)}_j)_{1,q_r} \end{matrix}\right]$$

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1}\cdots\int_{L_r} \phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots\xi_r)\, z_1^{\xi_1}\ldots z_r^{\xi_r}\, d\xi_1\ldots d\xi_r \tag{1.8}$$

where

$$\phi_i(\xi_i) = \frac{\prod\limits_{j=1}^{m_i}\Gamma(d^{(i)}_j-\delta^{(i)}_j\xi_i)\ \prod\limits_{j=1}^{n_i}\Gamma(1-c^{(i)}_j+\gamma^{(i)}_j\xi_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma(1-d^{(i)}_j+\delta^{(i)}_j\xi_i)\ \prod\limits_{j=n_i+1}^{p_i}\Gamma(c^{(i)}_j-\gamma^{(i)}_j\xi_i)} \tag{1.9}$$

$$\psi(\xi_1,\ldots,\xi_r) = \frac{\prod\limits_{j=1}^{n}\Gamma(1-a_j+\sum\limits_{i=1}^{r}\alpha^{(i)}_j\xi_i)}{\prod\limits_{j=1}^{q}\Gamma(1-b_j+\sum\limits_{i=1}^{r}\beta^{(i)}_j\xi_i)\ \prod\limits_{j=n+1}^{p}\Gamma(a_j-\sum\limits_{i=1}^{r}\alpha^{(i)}_j\xi_i)},\ \forall i\in\{1,\ldots,r\} \tag{1.10}$$

It is assumed that the various H-functions of several variables occurring in the paper always satisfy the appropriate existence and convergence conditions corresponding appropriately to those recorded in the book by Srivastava et.al [9,pp.251-253,Eqn's (C.4)-(C.6)]. In case r=2, it reduce to the H-function of two variables [9, p.82, Eqn. (6.1.1)].

2. MAIN INTEGRAL FORMULAE

FIRST INTEGRAL

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1\text{-}xy}{(1\text{-}x)(1\text{-}y)}\; R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1'}\left(\frac{1-y}{1-xy}\right)^{\lambda_1'}\right]\; H_{P_1,Q_1\,:\,P_2,Q_2\,;P_3,Q_3}^{0,N_1\,:\,1,N_2;1,N_3}\left[\begin{matrix} z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1} \\ \\ z_2\left(\frac{1-x}{1-xy}y\right)^{\mu_2}\left(\frac{1-y}{1-xy}\right)^{\nu_2}\end{matrix}\right]dxdy$$

$$=\sum_{v,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\frac{\theta_1^*(v,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H_{P_1+2,Q_1+1:P_2,Q_2;P_3,Q_3;1,1}^{0,N_1+2\;:\,1,N_2\;;1,N_3;1,1}$$

$$\left[\begin{matrix} z_1 \\ \\ z_2 \\ \frac{c}{d}\end{matrix}\left|\begin{matrix} T_1\,:(c_j,\gamma_j)_{1,P_2}\;;(e_j,E_j)_{1,P_3};(1-\nu+\delta n,1) \\ \\ T_2\,:(d_j,\delta_j)_{1,Q_2}\;;(f_j,F_j)_{1,Q_3};(0,1)\end{matrix}\right.\right]$$

(2.1)

Where θ_1^* is given by (1.4) and

(i)

$$T_1\equiv(1-\alpha_1-\xi_1R'\text{-}\xi_1'k;\mu_1,\mu_2,\xi_1q),(1-\beta_1-\lambda_1R'-\lambda_1'k;\nu_1,\nu_2,\lambda_1q),(a_j;\alpha_j,A_j,0)_{1,P_1}$$

$$T_2\equiv(b_j;\beta_j,B_j,0)_{1,Q_1},(1-\alpha_1-\beta_1-(\xi_1+\lambda_1)R'-(\xi_1'+\lambda_1')k;\mu_1+\nu_1,\mu_2+\nu_2,\xi_1q+\lambda_1q)$$

(2.2)

(ii) $\alpha_1,\beta_1,\mu_1,\mu_2,\nu_1,\nu_2$ are all positive, and

$$\text{Re}(\alpha_1)\;+\;\mu_1[\text{Re}\,(d_j/\delta_j)]+\;\mu_2[\text{Re}\,(f_j/F_j)]>0,$$

(2.3)

$$\text{Re}(\beta_1)\;+\;\nu_1[\text{Re}\,(d_j/\delta_j)]+\;\nu_2[\text{Re}\,(f_j/F_j)]>0.$$

(iii) Min $\{\xi_1, \lambda_1, \xi_1', \lambda_1', \mu_1, \nu_1\} \geq 0$, (not all zero simultaneously) (2.4)

(iv) $\left|\frac{c}{d}\right| < 1$ (2.5)

SECOND INTEGRAL

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\, R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1'}\left(\frac{1-y}{1-xy}\right)^{\lambda_1'}\right] H_{P_2,Q_2}^{1,N_2}\left[z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\right]dxdy$$

$$= \sum_{\nu,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\frac{\theta_1^*(\nu,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H_{2,1:P_2,Q_2;1,1}^{0,2:1,N_2;1,1}\left[\begin{array}{c|ll} z_1 & T_1^* : (c_j,\gamma_j)_{1,P_2} & ;(1-\nu+\delta n,1) \\ \frac{c}{d} & T_2^* : (d_j,\delta_j)_{1,Q_2} & ;(0,1)\end{array}\right] \quad (2.6)$$

Where

$$T_1^* \equiv (1-\alpha_1-\xi_1R'-\xi_1'k;\mu_1,\xi_1q),(1-\beta_1-\lambda_1R'-\lambda_1'k;\nu_1,\lambda_1q)$$

$$T_2^* \equiv (1-\alpha_1-\beta_1-(\xi_1+\lambda_1)R'-(\xi_1'+\lambda_1')k;\mu_1+\nu_1,\xi_1q+\lambda_1q) \quad (2.7)$$

The conditions of validity of (2.6) easily follow from those given in (2.1).

Proof: To evaluate the integral formula (2.1) we express $R_n^{(\alpha,\beta)}[x]$ and $S_n^m[x]$ in its series form with help of (1.3) and (1.7) respectively, change the order of integration and summation and put the value of $H[z_1,z_2]$ in terms of Mellin-Barnes contour integral by the application of (1.8).Next, we express the binomial terms obtained in the process in their contour integral form [9, p.18, Eq.(2.6.4)] and change the order of integration, we have :

$$= \sum_{\nu,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\frac{\theta_1^*(\nu,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$\left[\left(\frac{1}{2\pi i}\right)^3\int_{L_1}\int_{L_2}\int_{L_3}\phi(\xi,\eta)\theta_1(\xi)\theta_2(\eta)\frac{\Gamma(\nu-\delta n+s)}{\Gamma(s+1)}\left\{\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1+\xi_1R'+\xi_1'k+\mu_1\xi+\mu_2\eta+\xi_1qs}\right.\right.$$

$$\left(\frac{1-y}{1-xy}\right)^{\beta_1+\lambda_1 R'+\lambda_1' k+\nu_1\xi+\nu_2\eta+\lambda_1 qs)} \frac{1-xy}{(1-x)(1-y)}dxdy\Bigg\} z_1^{\xi} z_2^{\eta}\left(\frac{c}{d}\right)^{s} d\xi d\eta ds\Bigg] \tag{2.8}$$

Further, integrate the double integral in (2.8) with the help of the following result [2, p.415]:

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha}\left(\frac{1-y}{1-xy}\right)^{\beta} \frac{\text{1-xy}}{\text{(1-x)(1-y)}}\ \text{dx dy} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.9}$$

where Re (α) > 0, Re (β) > 0.

Finally, interpreting the $\xi,\eta\ \&\ s$ contour integral thus obtains in terms of the H-function of three variables [9, p.251, Eqn. (C.1)], we arrive at the right hand side of (2.1). The derivation of the formula (2.6) is similar to (2.1).

3. SPECIAL CASES OF MAIN INTEGRALS

On account of the most general nature of H-function of two and one variables, $R_n^{(\alpha,\beta)}(x)$ and $S_n^m[x]$ occurring in our formulae given by (2.1) and (2.6), a large number of integrals involving simpler functions of one and two variables can be easily obtained as their special cases. We however gave here only eight special cases by way of illustration:

(i) By applying our results given in (2.1) and (2.6) to the case of Hermite polynomial [10] and [12] and by setting

$$S_{n_1}^{2}[x]\to x^{n_1/2}H_{n_1}\left[\frac{1}{2\sqrt{x}}\right],$$

in which case $m_1=2, A_{n_1,k}=(-1)^k$,we have the following interesting consequences of the main results.

(i)

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1} \frac{\text{1-xy}}{\text{(1-x)(1-y)}}$$

$$R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]\left(\frac{1-x}{1-xy}y\right)^{\frac{\xi_1' n_1}{2}}\left(\frac{1-y}{1-xy}\right)^{\frac{\lambda_1' n_1}{2}}$$

$$H_{n_1}\left[\frac{1}{2\sqrt{\left(\frac{1-x}{1-xy}y\right)^{\xi_1'}\left(\frac{1-y}{1-xy}\right)^{\lambda_1'}}}\right] H^{1,N_1:1,N_2;1,N_3}_{P_1,Q_1:P_2,Q_2;P_3,Q_3}\left[\begin{matrix} z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1} \\ z_2\left(\frac{1-x}{1-xy}y\right)^{\mu_2}\left(\frac{1-y}{1-xy}\right)^{\nu_2}\end{matrix}\right]dxdy$$

$$=\sum_{v,u,t,e}\ \sum_{\text{k}=0}^{[\text{n}_1/\text{m}_1]}\frac{(-\text{n}_1)_{2\text{k}}(-1)^k}{\text{k}!}\ \frac{\theta_1^*(v,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H^{0,N_1+2:1,N_2;1,N_3;1,1}_{P_1+2,Q_1+1:P_2,Q_2;P_3,Q_3;1,1}\left[\begin{array}{c}z_1\\z_2\\ \\ \frac{c}{d}\end{array}\left|\begin{array}{l}T_1:(c_j,\gamma_j)_{1,P_2};(e_j,E_j)_{1,P_3};(1-\nu+\delta n,1)\\ \\ T_2:(d_j,\delta_j)_{1,Q_2};(f_j,F_j)_{1,Q_3};(0,1)\end{array}\right.\right] \tag{3.1}$$

The conditions of validity of (3.1) easily follow from those given in (2.1).

(ii)

$$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\,R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]$$

$$\left(\frac{1-x}{1-xy}y\right)^{\frac{\xi_1' n_1}{2}}\left(\frac{1-y}{1-xy}\right)^{\frac{\lambda_1' n_1}{2}}H_{n_1}\left[\frac{1}{2\sqrt{\left(\frac{1-x}{1-xy}y\right)^{\xi_1'}\left(\frac{1-y}{1-xy}\right)^{\lambda_1'}}}\right]$$

$$H^{1,N_2}_{P_2,Q_2}\left[z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\right]dxdy$$

$$=\sum_{v,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{2k}(-1)^k}{k!}\frac{\theta_1^*(v,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H^{0,2:1,N_2;1,1}_{2,1:P_2,Q_2;1,1}\left[\begin{array}{c}z_1\\ \frac{c}{d}\end{array}\left|\begin{array}{l}T_1^*:(c_j,\gamma_j)_{1,P_2};(1-\nu+\delta n,1)\\ \\ T_2^*:(d_j,\delta_j)_{1,Q_2};(0,1)\end{array}\right.\right] \tag{3.2}$$

The conditions of validity of (3.2) easily follow from those given in (2.6).

(2) For the Laguerre polynomials ([10] and [12]) setting $S'_{n_1}(x)\to L_{n_1}^{(\alpha')}$ in which case $m_1=1, A_{n_1,k}=\binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_k}$ and the results (2.1) and (2.6) reduce to the following formulae:

(iii)

$$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\,R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]$$

$$L_{n_1}^{\alpha'_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi'_1}\left(\frac{1-y}{1-xy}\right)^{\lambda'_1}\right] H^{0,N_1:1,N_2;1,N_3}_{P_1,Q_1:P_2,Q_2;P_3,Q_3}\left[\begin{matrix} z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1} \\ z_2\left(\frac{1-x}{1-xy}y\right)^{\mu_2}\left(\frac{1-y}{1-xy}\right)^{\nu_2}\end{matrix}\right]dxdy$$

$$=\sum_{v,u,t,e}\sum_{k=0}^{[n_1]}\frac{(-n_1)_k}{k!}\binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_k}\frac{\theta_1^*(v,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H^{0,N_1+2\;:\;1,N_2\;;\;1,N_3;1,1}_{P_1+2,Q_1+1:P_2,Q_2;P_3,Q_3;1,1}\left[\begin{matrix} z_1 \\ z_2 \\ \frac{c}{d}\end{matrix}\left|\begin{matrix} T_1\;:(c_j,\gamma_j)_{1,P_2}\;;(e_j,E_j)_{1,P_3};(1-\nu+\delta n,1) \\ T_2\;:(d_j,\delta_j)_{1,Q_2}\;;(f_j,F_j)_{1,Q_3};(0,1)\end{matrix}\right.\right] \tag{3.3}$$

The conditions of validity of (3.3) easily follow from those given in (2.1).

(iv)

$$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\,R_n^{(\alpha,\beta)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1};a,b,c,d;p,q;\gamma,\delta;1\right]$$

$$L_{n_1}^{\alpha'}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi'_1}\left(\frac{1-y}{1-xy}\right)^{\lambda'_1}\right] H^{1,N_2}_{P_2,Q_2}\left[z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\right]dxdy$$

$$=\sum_{v,u,t,e}\sum_{k=0}^{[n_1]}\frac{(-n_1)_k}{k!}\binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_k}\frac{\theta_1^*(v,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H^{0,2:1,N_2;1,1}_{2,1:P_2,Q_2;1,1}\left[\begin{matrix} z_1 \\ \frac{c}{d}\end{matrix}\left|\begin{matrix} T_1^*\;:(c_j,\gamma_j)_{1,P_2}\;;(1-\nu+\delta n,1) \\ T_2^*\;:(d_j,\delta_j)_{1,Q_2}\;;(0,1)\end{matrix}\right.\right]$$

(3.4)

The conditions of validity of (3.4) easily follow from those given in (2.6).

(3) For Jacobi polynomials ([4] and [7]) setting $R_n^{(\alpha,\beta)}[x] \rightarrow P_n^{(\beta,\alpha)}[x]$ in which case $a=b=d=p=q=\gamma=\delta=1, c=-1, k_n' = \frac{2^n n!}{(-1)^n}$ and the results (2.1) and (2.6) reduce to the following formulae:

(v)

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1} \frac{1\text{-xy}}{(1\text{-x})(1\text{-y})}\, \mathrm{P}_n^{(\beta,\alpha)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1}\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1'}\left(\frac{1-y}{1-xy}\right)^{\lambda_1'}\right] \mathrm{H}^{0,\mathrm{N}_1\,:\,1,\mathrm{N}_2;1,\mathrm{N}_3}_{\mathrm{P}_1,\mathrm{Q}_1\,:\,\mathrm{P}_2,\mathrm{Q}_2;\mathrm{P}_3,\mathrm{Q}_3}\left[\begin{matrix} z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1} \\ z_2\left(\frac{1-x}{1-xy}y\right)^{\mu_2}\left(\frac{1-y}{1-xy}\right)^{\nu_2}\end{matrix}\right] dxdy$$

$$= \sum_{\nu=0}^{n}\sum_{t=0}^{\infty}\sum_{\mathrm{k}=0}^{[\mathrm{n}_1/\mathrm{m}_1]} \frac{(-\mathrm{n}_1)_{\mathrm{m}_1\mathrm{k}}}{\mathrm{k}!}\,\mathrm{A}_{\mathrm{n}_1,\mathrm{k}}\,\frac{(1+\beta)_n\,\theta_1^*(\nu,t)}{n!\,\Gamma(\nu-\delta n)}$$

$$\mathrm{H}^{0,\mathrm{N}_1+2\,:\,1,\mathrm{N}_2\,;\,1,\mathrm{N}_3;1,1}_{\mathrm{P}_1+2,\mathrm{Q}_1+1:\mathrm{P}_2,\mathrm{Q}_2;\mathrm{P}_3,\mathrm{Q}_3;1,1}\left[\begin{matrix} z_1 \\ z_2 \\ \frac{c}{d}\end{matrix}\left|\begin{matrix}\mathrm{T}_1 : (\mathrm{c_j},\gamma_j)_{1,\mathrm{P}_2}\,;(e_j,E_j)_{1,\mathrm{P}_3};(1-\nu+\delta n,1) \\ \mathrm{T}_2 : (\mathrm{d_j},\delta_j)_{1,\mathrm{Q}_2}\,;(f_j,F_j)_{1,\mathrm{Q}_3};(0,1)\end{matrix}\right.\right]$$

(3.5)

Where

$$\theta_1^*(\nu,t) = \frac{(-1)^t(-n)_\nu(\nu)_t(1+\alpha+\beta+n)_\nu}{t!\,2^\nu(1+\beta)_\nu}$$

The conditions of validity of (3.5) easily follow from those given in (2.1).

(vi)

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\,P_n^{(\beta,\alpha)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1}\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi'_1}\left(\frac{1-y}{1-xy}\right)^{\lambda'_1}\right]\; H_{P_2,Q_2}^{1,N_2}\left[z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\right]dxdy$$

$$=\sum_{\nu=0}^{n}\sum_{t=0}^{\infty}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\,\frac{(1+\beta)_n\,\theta_1^*(\nu,t)}{n!\,\Gamma(\nu-\delta n)}$$

$$H_{2,1:P_2,Q_2;1,1}^{0,2:1,N_2;1,1}\left[\begin{array}{c|ll} z_1 & T_1^*:(c_j,\gamma_j)_{1,P_2} & ;(1-\nu+\delta n,1)\\ \frac{c}{d} & T_2^*:(d_j,\delta_j)_{1,Q_2} & ;(0,1)\end{array}\right] \tag{3.6}$$

The conditions of validity of (3.6) easily follow from those given in (2.6).
(4) For generalized polynomial set ([5] and [6]) setting $R_n^{(\alpha,\beta)}[x]\rightarrow S_n^{(\alpha,\beta,\tau)}[x]$ in which case $p=d=1, c=-\tau, k'_n=1, \beta\rightarrow\beta/\tau$ and the results (2.1) and (2.6) reduce to the following formulae:

(vii)

$$\int_0^1\int_0^1 \left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\,S_n^{(\alpha,\beta,\tau)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1}\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi'_1}\left(\frac{1-y}{1-xy}\right)^{\lambda'_1}\right]\; H_{P_1,Q_1:P_2,Q_2;P_3,Q_3}^{0,N_1:1,N_2;1,N_3}\left[\begin{array}{c} z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\\ z_2\left(\frac{1-x}{1-xy}y\right)^{\mu_2}\left(\frac{1-y}{1-xy}\right)^{\nu_2}\end{array}\right]dxdy$$

$$=\sum_{\nu,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\,\frac{\theta_1^{**}(\nu,u,t,e)}{\Gamma(\nu-\delta n)}$$

$$H^{0,N_1+2\ :1,N_2\ ;1,N_3;1,1}_{P_1+2,Q_1+1:P_2,Q_2;P_3,Q_3;1,1}\left[\begin{array}{c|l} z_1 & T_1:(c_j,\gamma_j)_{1,P_2}\ ;(e_j,E_j)_{1,P_3};(1-\nu+\delta n,1) \\ z_2 & T_2:(d_j,\delta_j)_{1,Q_2}\ ;(f_j,F_j)_{1,Q_3};(0,1) \\ \frac{c}{d} & \end{array}\right]$$

(3.7)

Where

$$\theta_1^{**}(v,u,t,e)=\frac{b^{\gamma n}k^n(-1)^t(-v)_u(-t)_e}{v!u!t!e!}(\alpha)_t(-\tau)^v\left(\frac{a}{b}\right)^t$$
$$\frac{(-\alpha-\gamma n)_e}{(1-\alpha-t)_e}(-\beta/\tau-\delta n)_v\left(\frac{e+\lambda+qu}{k}\right)_n.R'=kn+qv+t,\left(\frac{pe+\lambda+qu}{k}\right)_n.$$

(3.8)

The conditions of validity of (3.7) easily follow from those given in (2.1).

(viii)

$$\int_0^1\int_0^1\left(\frac{1-x}{1-xy}y\right)^{\alpha_1}\left(\frac{1-y}{1-xy}\right)^{\beta_1}\frac{1-xy}{(1-x)(1-y)}\ S_n^{(\alpha,\beta,\tau)}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi_1}\left(\frac{1-y}{1-xy}\right)^{\lambda_1}\right]$$

$$S_{n_1}^{m_1}\left[\left(\frac{1-x}{1-xy}y\right)^{\xi'_1}\left(\frac{1-y}{1-xy}\right)^{\lambda'_1}\right]\ H^{1,N_2}_{P_2,Q_2}\left[z_1\left(\frac{1-x}{1-xy}y\right)^{\mu_1}\left(\frac{1-y}{1-xy}\right)^{\nu_1}\right]dxdy$$
$$=\sum_{v,u,t,e}\sum_{k=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k}}{k!}A_{n_1,k}\ \frac{\theta_1^{**}(v,u,t,e)}{\Gamma(\nu-\delta n)}$$
$$H^{0,2:1,N_2;1,1}_{2,1:P_2,Q_2;1,1}\left[\begin{array}{c|l} z_1 & T_1^*:(c_j,\gamma_j)_{1,P_2}\ ;(1-\nu+\delta n,1) \\ \frac{c}{d} & \\ & T_2^*:(d_j,\delta_j)_{1,Q_2}\ ;(0,1) \end{array}\right]$$

(3.9)

The conditions of validity of (3.9) easily follow from those given in (2.6).

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Praveen Agrawal
Department of Mathematics,
Anand International College of Engineering,
Jaipur-303012,
India

Shilpi Jain
Deparment of Mathematics,
Poornima College of Engineering,
Jaipur-302017,
India
Email: goyal_praveen2000@rediffmail.com

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