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NUMERICAL SOLUTION OF SAWADA- KOTRA EQUATION BY HOMOTOPY ANALYSIS METHOD[1]

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Abstract

In this article, homotopy analysis method is applied to solve well-known nonlinear partial differential equation, called Sawada- Kotra (SK). Numerical solutions obtained by the homotopy analysis method are compared with the exact solutions. The results reveal that the method is very effective and simple.

Mathematics Subject Classification 2000: 35Q53; 35B45

General Terms: Homotopy analysis method; Sawada- Kotra equation.

1. INTRODUCTION

The homotopy analysis method has been developed by Liao [5, 9] to obtain series solutions of controllable convergence to various nonlinear problems Recently, a great deal of interest has been focused on the applications of the homotopy analysis method, well addressed in [1, 2, 4-8, 11].

2. PRIMARY HEAD

To describe the basic ideas the homotopy analysis method, we consider the following differential equation,

$$N(u(r,t)]=0, \tag{1}$$

where N is a nonlinear operator, r and t are independent variables, $u(r,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called $zero\text{-}order$ deformation equation

$$(1-p)L\left(\varphi(r,t;p)-u_0(r,t)\right)=p\hbar H(r,t)N(r,t;p), \tag{2}$$

where $p\in[0,1]$ is the embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, $\varphi(r,t;p)$ is a unknown function on independent variables $r,t,p.$ It is important that one has great freedom to choose auxiliary parameter $\hbar$ in homotopy analysis method. If $p=0$ and $p=1$, it holds

$$\varphi(r,t;0)=u_0(r,t),\qquad \varphi(r,t;1)=u(r,t),\tag{3}$$

Thus, as p increases from 0 to 1, the solution $\varphi(r,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\varphi(r,t;p)$, in Taylor series with respect to p, we have

$$\varphi(r,t;p)=u_0(r,t)+\sum_{m=1}^{\infty}u_m(r,t)p^m,\tag{4}$$

where

$$u_m(r,t)=\frac{1}{m!}\frac{\partial^m\varphi(r,t;p)}{\partial p^m}\Big|_{p=0}.\tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at p = 1, then we have

$$u(r,t)=u_0(r,t)+\sum_{m=1}^{\infty}u_m(r,t).\tag{6}$$

Define the vector $\overrightarrow{u_n}=\{u_0,u_1,\ldots u_n\}$.

Differentiating equation (2) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $m!$, we obtain the $mth-order$ deformation equation

$$L[u_m-\chi_m u_{m-1}]=\hbar H(r,t)R_m(\vec{u}_{m-1}),\tag{7}$$

where

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N(r,t;p)}{\partial p^{m-1}}\Big|_{p=0},\tag{8}$$

and

$$\chi_m=\begin{cases}0 & m\le 1,\\ 1 & m>1.\end{cases}\tag{9}$$

Applying L^{-1} on both side of equation (7), we get

$$u_m(r,t)=\chi_m u_{m-1}(r,t)+\hbar L^{-1}[H(r,t)R_m(\vec{u}_{m-1})].\tag{10}$$

In this way, it is easily to obtain u_m form $m\ge 1$, , at $Mth-order,$ we have

$$u(r,t)=\sum_{m=0}^{M}u_m(r,t).\tag{11}$$

When $M\to\infty.$ we get an accurate approximation of the original equation (1). For the convergence of the above method we refer the reader to Liao [2]. If equation (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the homotopy analysis method will give a solution among many other (possible) solutions.

Example : Sawada-Kotra equation [3]

$$\frac{\partial u}{\partial t}+45u^2\frac{\partial u}{\partial x}-15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}-15u\frac{\partial^3 u}{\partial x^3}+\frac{\partial^5 u}{\partial x^5}=0. \qquad (12)$$

With initial condition,

$$u(x,0)=\ a_0 - 2c_2\sec h^2(\sqrt{c_2}\,x). \qquad (13)$$

With the exact solution

$$u=a_0-2c_2\sec h^2(\sqrt{c_2}\,(x-(45a_0^2-60a_0c_2+16c_2^2)t)),\quad c_2>0.$$

Where a_0, c_2 are arbitrary constants.

To solve the equation (12) by means of homotopy analysis method, according to the initial conditions denoted in equation (13), it is natural to choose

$$u_0=a_0-2c_2\sec h^2(\sqrt{c_2}\,(x-(45a_0^2-60a_0c_2+16c_2^2)t)). \qquad (14)$$

We choose the linear operator

$$L(\varphi(x,t;p)]=\frac{\partial\varphi_1(x,t;p)}{\partial t},$$

with the property $L[c]=0.$ Where *c* is constant. We now define a nonlinear operator as

$$N[\varphi(x,t;p)]=\frac{\partial\varphi(x,t;p)}{\partial t}+45\varphi(x,t;p)^2\frac{\partial\varphi(x,t;p)}{\partial x}$$

$$-15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}-15\varphi(x,t;p)\frac{\partial^3\varphi(x,t;p)}{\partial x^3}+\frac{\partial^5\varphi(x,t;p)}{\partial x^5}.$$

Using above definition, with assumption $H(x,t)=1.$ We construct the $\text{zeroth}-\text{order}$ deformation equations

$$(1-p)L\left(\varphi(x,t;p)-u_0(x,t)\right)=p\hbar N(\varphi(x,t;p)),$$

obviously, when $p=0$ and $p=1,$

$$\varphi(x,t;0)=u_0(x,t),\quad \varphi(x,t;0)=u(x,t),$$

Thus, we obtain the $mth-order$ deformation equation

$$L[u_m-\chi_m u_{m-1}]=\hbar R_m(\vec{u}_{m-1}). \qquad (15)$$

Where

$$R_m(\vec{u}_{m-1})=\frac{\partial U_{m-1}}{\partial t}+45\sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1}U_iU_k\frac{\partial U_{m-k-i-1}}{\partial x}$$

$$-15\sum_{k=0}^{m-1}\frac{\partial U_k}{\partial x}\frac{\partial^2 U_{m-1-k}}{\partial x^2}-15\sum_{k=0}^{m-1}U_k\frac{\partial^3 U_{m-1-k}}{\partial x^3}+\frac{\partial^5 U_m}{\partial x^5}.$$

Now, the solution of the $mth-order$ order deformation equation (15)

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar L^{-1}[R_m(\vec{u}_{m-1})], \qquad (16)$$

Finally, we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t).$$

From equations (14) and (16) and subject to initial condition

$$u_m(x,0)=0,\quad m\geq 1.$$

We obtain

$$u_0 = a_0 - 2c_2 \sec h^2(\sqrt{c_2}x).$$

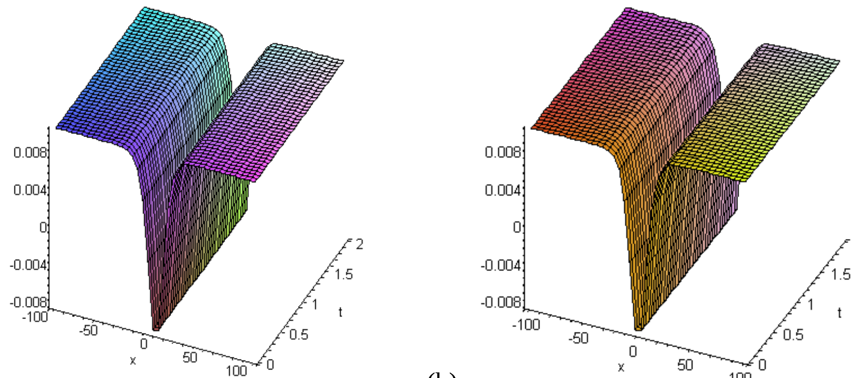
$$\begin{aligned}
u_1 = &-\hbar 720 t a_0\sqrt{c_2^5}\sec h^2(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x) \\
&+\hbar 480 t a_0\sqrt{c_2^5}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)+\hbar 2160 t\sqrt{c_2^7}\sec h^4(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x) \\
&-\hbar 1200 t\sqrt{c_2^7}\sec h^4(\sqrt{c_2}x)\tanh(\sqrt{c_2}x)+\hbar 180\sqrt{c_2^3}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) t a_0^2 \\
&-\hbar 720\sqrt{c_2^5}\sec h^4(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) t a_0+\hbar 720\sqrt{c_2^7}\sec h^6(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) t \\
&+\hbar 1440\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh^5(\sqrt{c_2}x) t-\hbar 1920\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh^3(\sqrt{c_2}x) t \\
&+\hbar 544\sqrt{c_2^7}\sec h^2(\sqrt{c_2}x)\tanh(\sqrt{c_2}x) t,
\end{aligned}$$

Suppose $u^* = \sum_{j=0}^{3} U_j$, the results are presented in Table 1 and Fig. 1.

Table 1

The numerical results, when $a_0 = c_2 = 0.01$ and $h = -1$ for solutions of Eq. (10) for initial condition (11).

x	t	$u^*(x,t)$	$\lvert u^* - u\rvert$
0.1	0.1	-0.00999800053325245730	1.0×10^{-19}
0.1	0.15	-0.00999800073320080290	1.0×10^{-19}
0.15	0.1	-0.00999550127471400001	1×10^{-20}
0.2	0.3	-0.00999200453139060139	1×10^{-20}
0.35	0.25	-0.00997552348861796656	4×10^{-20}
0.45	0.45	-0.00995956269009242584	4×10^{-20}
0.5	0.3	-0.00995008919529557727	3×10^{-20}
0.45	0.7	-0.00995956717739295321	1×10^{-20}
0.4	0.8	-0.00996804687387015724	4×10^{-20}
0.8	0.8	-0.00987256954075552987	3×10^{-20}
0.85	0.95	-0.00985622515843887668	8×10^{-20}
0.9	0.9	-0.00983890285245324707	7×10^{-20}
0.95	0.9	-0.00982061427034014788	2×10^{-20}
1	1	-0.00980136528619636983	3×10^{-20}



(a) (b)

Fig. 1. The numerical results for $u^*(x,t)$, are, respectively (a) in comparison with the analytical solutions $u(x,t)$ are, respectively (b) with the initial condition (11) of Eq (10). when , $a_0 = .01$, and $c_2 = .01$ and $h = -1$.

3. CONCLUSION

Homotopy analysis method has been successfully applied to find the solution of non-linear Sawada-Kotra (SK) equation is presented in Table 1, for differential results of x, t to show the stability of the method. The approximate solutions obtained by the homotopy analysis method are compared with exact solutions. It can be concluded that the homotopy analysis method is very powerful and efficient technique in finding exact solutions for wide classes of problems. In our work, we use the maple package to carry the computations.

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