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SUBORDINATION AND SUPERORDINATION RESULTS INVOLVING CERTAIN ANALYTIC FUNCTIONS

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Abstract

The main purpose of this paper is to derive some subordination and superordination results involving certain class of integral operators for analytic functions. Several sandwich results are also obtained.

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1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic function in the unit disk

$$\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (n \in \mathbb{N} := \{1, 2, \ldots\}). \tag{1.1}$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (z \in \Delta). \tag{1.2}$$

With a view to recalling the principle of subordination between analytic functions, let f and g are *analytic* and there exists a *Schwarz* function $w(z)$, analytic in the unit disk Δ, with

$$w(0) = 0, \ \ |w(z)| < 1 \quad (z \in \Delta), \tag{1.3}$$

such that $f(z) = g(w(z))$, then the function f is called *subordinate* to g and defined by

$$f \prec g \text{ or } f(z) \prec g(z) \qquad (z \in \Delta). \tag{1.4}$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta). \tag{1.5}$$

Suppose that h and k are analytic functions in Δ and $\phi(r, s, t; z) : \mathcal{C}^3 \times \Delta \longrightarrow \mathcal{C}$. If h and $\phi(h(z), zh'(z), z^2h''(z); z)$ are univalent and if h satisfies the second - order superodination

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z), \tag{1.6}$$

then h is the solution of the differential superordination (1.6). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called *subordinant* if $q \prec h$ for all h satisfying (1.6). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinats q of (1.6) is said to be *best subordinant*. Miller and Mocanu [12] have obtained conditions on k, q and ϕ for which the following implication holds:

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z) \Longrightarrow q(z) \prec h(z). \tag{1.7}$$

Ali *et al.* [1] obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.8}$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = q_2(0) = 1$.

Recently, Shanmugam *et al.* [13, 14] and Goyal *et al.* [7] also obtained sandwich results for certain classes of analytic functions. Further subordination results can be found in [15] and [16].

In a recent paper, Li and Srivastava [10] introduced and studied the function class Φ defined by

$$\Phi := \left\{ \lambda(t) : \lambda(t) \geq 0 \;\; (0 \leq t \leq 1) \text{ and } \int_0^1 \lambda(t)dt = 1 \right\} \tag{1.9}$$

Fournier and Ruscheweyh ([6], see also [10]) have considered an integral operator which involves a non negative function

$$\lambda : [0,1] \to R \; s.t. \int_0^1 \lambda_\alpha(t)dt = 1.$$

Many applications of the real valued function $\lambda(t)$ depends also upon a suitable parameter α. We thus consider the Fournier - Ruscheweyh integral operator in the following modified form [10, p. 131, eq. (2.2)]:

$$\mathcal{I}_\lambda^\alpha f(z) := \int_0^1 \lambda_\alpha(t) \frac{f(tz)}{t} dt \qquad\qquad (f \in \mathcal{A}), \tag{1.10}$$

where the real-valued function λ_α and $\lambda_{\alpha-1}$ satisfy the conditions :

(1) For a suitable parameter α,

$$\lambda_{\alpha-1}(t) \in \Phi, \;\; \lambda_\alpha(t) \in \Phi \text{ and } \lambda_\alpha(1) = 0, \tag{1.11}$$

(2) There exists a constants c $(-1 < c \leq 2)$ such that

$$c\lambda_\alpha(t) - t\lambda_\alpha^{'}(t) = (c+1)\lambda_{\alpha-1}(t) \qquad (0 < t < 1; -1 < c \leq 2). \tag{1.12}$$

For $\mathcal{I}_\lambda^\alpha$ operator, under the conditions (1.11) and (1.12) (see e.g. [10, p. 132, eq. (2.6)]), we have

$$z(\mathcal{I}_\lambda^\alpha f(z))' = -c\mathcal{I}_\lambda^\alpha f(z) + (c+1)\,\mathcal{I}_\lambda^{\alpha-1} f(z) \tag{1.13}$$

The main object of this paper is to apply a method based on the differential subordination in order to derive several subordination, superordination and sandwich results.

2. Definitions and Preliminaries

In order to prove our main results, we need the following definition and lemmas.
Definition 2.1([12]) :*Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\bar{\Delta} - E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial\triangle : \lim_{z\to\zeta} f(z) = \infty \right\} \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\triangle - E(f)$.

We shall require certain results due to Miller and Mocanu ([11] and [12]), Bulboacã [3] and Shanmugam *et al.* [14].
Lemma 2.2 ([11] , Theorem 3.4h, p.132) *Let $q(z)$ be univalent in the unit disk $\triangle$ and let θ and ϕ be analytic in the domain D containing $q(\triangle)$ with $\phi(w) \neq 0$ when $w \in q(\triangle)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in $\triangle$;*

(2) $\Re\,(zh'(z)/Q(z)) > 0$ *for* $z \in \Delta$.

If $p(z)$ is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p \prec q$ and q is the best dominant.
Lemma 2.3 ([14]). *Let $q(z)$ be a convex univalent function in Δ and $\psi, \gamma \in \mathcal{C}$ with $Re\,(1 + (zq''(z)/q'(z))) > max\,\{0, -Re(\psi/\gamma)\}$. If $p(z)$ is analytic in Δ and*

$$\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z), \tag{2.3}$$

then $p \prec q$ and q is the best dominant.
Lemma 2.4 ([3]). *Let q be convex univalent in the unit disk Δ and let ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) *$\Re\,[\vartheta'(q(z))/\varphi(q(z))] > 0$ for $z \in \Delta$;*

(2) *$zq'(z)\varphi(q(z))$ is starlike univalent in $z \in \Delta$.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.4}$$

then $q \prec p$ and q is the best subordinant.
Lemma 2.5 ([12]). *Let q be convex univalent in Δ and $\gamma \in \mathcal{C}$. Further assume that $\Re(\gamma) > 0$. If $p \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \gamma zp'(z)$ is univalent in Δ, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z), \tag{2.5}$$

which implies that $q \prec p$ and q is the best subordinant.

3. Subordination Results

Theorem 3.1. *Let $q(z)$ be univalent in the unit disk $\triangle$, $0 \neq \beta \in \mathbb{C}$ and $0 < \mu < 1$. Further suppose that q satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\mu}{\beta}\right)\right\}, \tag{3.1}$$

and conditions (1.11) and (1.12) hold. If $f \in \mathcal{A}$ satisfies the subordination

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec q(z) + \frac{\beta}{\mu} zq'(z), \tag{3.2}$$

then

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec q(z), \tag{3.3}$$

and q is the best dominant.

Proof. Suppose that

$$h(z) := \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu. \tag{3.4}$$

Differentiating (3.4) with respect to z logarithmically, we get

$$\frac{zh'(z)}{h(z)} = \mu\left(1 - \frac{z[\mathcal{I}_\lambda^\alpha f(z)]'}{\mathcal{I}_\lambda^\alpha f(z)}\right). \tag{3.5}$$

Now, in view of (1.13), we obtain

$$\frac{zh'(z)}{h(z)} = \mu(1+c)\left(1 - \frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right) \tag{3.6}$$

which in the light of the hypothesis (3.2) of the theorem yields the following subordination

$$h(z) + \frac{\beta}{\mu} zh'(z) \prec q(z) + \frac{\beta}{\mu} zq'(z) \tag{3.7}$$

An application of Lemma 2.3 with $\gamma = \beta/\mu$ and $\psi = 1$, leads to (3.3).

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, we get the following result.

Corollary 3.2. *Let $-1 \leq B < A \leq 1$, $\beta \neq 0$ and*

$$\Re\left(\frac{1 - Bz}{1 + Bz}\right) > \max\left\{0,\ -\Re\left(\frac{\mu}{\beta}\right)\right\},$$

Also, the conditions (1.11) and (1.12) are satisfied. If $f \in \mathcal{A}$, and

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec \frac{1+Az}{1+Bz} + \frac{\beta}{\mu}\frac{(A-B)z}{(1+Bz)^2}, \tag{3.8}$$

then

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec \frac{1+Az}{1+Bz} \tag{3.9}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}, \quad 0 < \rho \leq 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.3. *Let* $0 < \rho \leq 1$ *and*

$$\Re\left(\frac{1+z^2+2\rho z}{1-z^2}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\beta}\right)\right\}$$

holds. Also, the conditions (1.11) and (1.12) are satisfied. If $f \in \mathcal{A}$, *and*

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu} - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu}$$

$$\prec \left(\frac{1+z}{1-z}\right)^{\rho} + \frac{\beta}{\mu}\left(\frac{2z}{1-z^2}\right)\left(\frac{1+z}{1-z}\right)^{\rho}, \tag{3.10}$$

then

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\rho} \tag{3.11}$$

and $\left(\frac{1+z}{1-z}\right)^{\rho}$ *is the best dominant.*

Theorem 3.4. *Let* q *be univalent in the unit disk* Δ *and* $f \in \mathcal{A}$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* Δ *and* q *satisfies*

$$\Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}q^2(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \qquad (z \in \Delta; \delta, \xi, \beta \in \mathbb{C}; \beta \neq 0). \tag{3.12}$$

Suppose also that the conditions (1.11) and (1.12) are satisfied and

$$\begin{aligned} &\psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z) \\ &= \gamma + \xi\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{\mu} + \delta\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{2\mu} \\ &\quad + \beta\mu(c+1)\left(1 - \frac{(1-\eta)\mathcal{I}_\lambda^{\alpha-1} f(z) + \eta\mathcal{I}_\lambda^\alpha f(z)}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right) \\ &\qquad (z \in \Delta; 0 \leq \eta \leq 1, 0 < \mu < 1). \end{aligned} \tag{3.13}$$

If

$$\psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z) \prec \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{3.14}$$

then

$$\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{\mu} \prec q(z), \tag{3.15}$$

and q *is the best dominant.*

Proof. Define a function $k(z)$ by

$$k(z) := \left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{\mu}. \tag{3.16}$$

Then a computation shows that

$$\frac{zk'(z)}{k(z)} = (c+1)\mu\left(1 - \frac{(1-\eta)\mathcal{I}_\lambda^{\alpha-1} f(z) + \eta\mathcal{I}_\lambda^\alpha f(z)}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right).$$

By setting

$$\theta(w) := \gamma + \xi w + \delta w^2 \text{ and } \phi(w) := \frac{\beta}{w},$$

it can be easily observed that $\theta(w)$ is an analytic in $\mathcal{C}$ and $\phi(w) \neq 0$ is an analytic in $\mathcal{C}\setminus\{0\}$. Moreover, we let

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)},$$

and

$$p(z) = \theta(q(z)) + Q(z) = \gamma + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}. \tag{3.17}$$

From (3.11), we find that $Q(z)$ is starlike univalent in the unit disk Δ. Also, from (3.17), we get

$$\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0.$$

An application of Lemma 2.2 to (3.14) yields the desired result.

Upon setting $q(z) = e^{Az}$ and $\eta = 1$ in Theorem 3.4, we get the following result.

Corollary 3.5. *Let $f \in \mathcal{A}$, and $0 < \mu < 1$. Further suppose that*

$$\Re\left(\beta + \xi e^{Az} + 2\delta e^{2Az}\right) > 0, \tag{3.18}$$

and the conditions (1.11) and (1.12) are satisfied. If

$$\gamma + \xi\left(\tfrac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu + \delta\left(\tfrac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{2\mu} + \beta\mu(c+1)\left(1 - \tfrac{\mathcal{I}_\lambda^\alpha f(z)}{\mathcal{I}_\lambda^{\alpha+1} f(z)}\right) \prec \gamma + \xi e^{Az} + \delta e^{2Az} + \beta A z,$$

then

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec e^{Az} \quad , \tag{3.20}$$

and e^{Az} is the best dominant.

4. Superordination Results

By Lemma 2.5, we obtain the following result.

Theorem 4.1. *Let $q(z)$ be convex univalent function in the unit disk Δ, and $0 < \mu < 1$. Suppose that $Re\,(\beta) > 0$ and $\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \in \mathcal{H}(q(0), 1) \cap Q$. Let*

$$(1 + \beta + \beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \tag{4.1}$$

be univalent in Δ and conditions (1.11) and (1.12) are satisfied. If $f \in \mathcal{A}$ satisfies

$$q(z) + \frac{\beta}{\mu}zq'(z) \prec (1 + \beta + \beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu, \tag{4.2}$$

then

$$q(z) \prec \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \quad , \tag{4.3}$$

and q is the best subordinant.

Proof. Let

$$p(z) := \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu. \tag{4.4}$$

Differentiating (4.4) logarithmically, we get

$$p(z) + \frac{\beta}{\mu} zp'(z) = (1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu$$

Theorem 4.1, follows by application of Lemma 2.5.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq A < B \leq 1)$ in Theorem 4.1, we easily get

Corollary 4.2. *Let $-1 \leq A < B \leq 1, \Re(\beta) > 0$ and $0 < \mu < 1$. Suppose that*

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \in \mathcal{H}(q(0), 1) \cap Q$$

and conditions (1.11) and (1.12) hold. If

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \tag{4.5}$$

be univalent in Δ and $f \in \mathcal{A}$ satisfies the superordination

$$\frac{\beta(A-B)z}{\alpha(1+Bz)^2} + \frac{1+Az}{1+Bz} \prec (1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu, \tag{4.6}$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \tag{4.7}$$

and $(1+Az)/(1+Bz)$ is the best subordinant.

Theorem 4.3. *Let q be convex univalent in the unit disk Δ , $0 < \mu < 1$ and $f \in \mathcal{A}$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in Δ and q satisfies*

$$\Re\left(1 + \tfrac{\xi}{\beta} q(z) + \tfrac{2\delta}{\beta}(q(z))^2\right) > 0 \qquad (z \in \Delta; \delta, \xi, \beta \in \mathbb{C};\ \beta \neq 0), \tag{4.8}$$

and

$$\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^\mu \in \mathcal{H}[q(0), 1] \cap Q.$$

Also suppose that $\psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z)$, is defined by (3.13), is univalent in the unit disk Δ and conditions (1.11) and (1.12) hold. If

$$\gamma + \xi q(z) + \delta(q(z))^2 + \beta \frac{zq'(z)}{q(z)} \prec \psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z), \tag{4.9}$$

then

$$q(z) \prec \left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^\mu \tag{4.10}$$

and q is the best subordinant.

Proof. By setting

$$\vartheta(w) := \gamma + \xi w + \delta w^2 \text{ and } \varphi(w) := \frac{\beta}{w}.$$

It is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\varphi(w) \neq 0$ is analytic in $\mathbb{C}\setminus\{0\}$. Since q is convex univalent function in Δ, it follows that

$$\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right) = \Re\left(\frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}q^2(z)\right) > 0.$$

The assertion (4.10) follows by an application of Lemma 2.4.

5. Sandwich Results

Combining the results of differential subordination and superordination, we state the following "sandwich results".

Theorem 5.1. *Let q_1 be convex univalent and let q_2 be univalent in Δ, and $\beta \in \mathbb{C}$. Suppose that $\Re(\beta) > 0$, $0 < \mu < 1$ and q_1 satisfies (4.1) and q_2 satisfies (3.1), respectively. If*

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \in \mathcal{H}(q(0),1) \cap Q$$

and

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu$$

be univalent in Δ and conditions (1.11) and (1.12) are satisfied. If $f \in \mathcal{A}$ satisfies

$$q_1(z) + \tfrac{\beta}{\mu} z q_1'(z) \prec (1+\beta+\beta c)\left(\tfrac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\tfrac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\tfrac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu$$
$$\prec q_2(z) + \tfrac{\beta}{\mu} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec q_2(z),$$

and q_1 and q_2 are the best subordinant and dominant, respectively.

Theorem 5.2. *Let q_1 be convex univalent and satisfies (4.7) and let q_2 be univalent and satisfies (3.13). Further suppose that $\delta, \beta, \xi \in \mathcal{C}$, $\beta \neq 0$, $0 < \mu < 1$ and $0 \leq \eta \leq 1$ and*

$$\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^\mu \in \mathcal{H}[q(0),1] \cap Q.$$

Let $\psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z)$, defined by (3.13), is univalent in the unit disk Δ and conditions (1.11) and (1.12) hold. If

$$\gamma + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{z{q_1}'(z)}{q_1(z)} \prec \psi(\lambda, \mu, \xi, \eta, \alpha, \beta, \gamma, \delta; z) \prec \gamma + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{z{q_2}'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^\mu \prec q_1(z),$$

and q_1 and q_2 are the best subordinant and dominant, respectively.

6. Further Applications and Observations

By assigning appropriate special values to the various parameters involved, we can derive several interesting applications of our results associated with many classes of linear operators. For the sake of illustration two useful choices of $\lambda_\alpha(t)$ will be considered here for Theorem 3.1.

Setting

$$\lambda_\alpha(t) = \frac{2^\alpha}{\Gamma(\alpha)} t \left(\log \frac{1}{t}\right)^{\alpha-1} \qquad (\alpha > 0)$$

in (1.10), we obtain the integral operator $\mathcal{P}^\alpha$ given by

$$\mathcal{P}^\alpha(t) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \qquad (f \in \mathcal{A};\ \alpha > 0).$$

The integral operator P^α is essentially the same as the multiplier transformation I^λ due to Flett [5] and studied subsequently by Li [8], Li and Srivastava [9] and many others.

In the case when $\alpha > 1$, the conditions (1.11) and (1.12) are satisfied for $c = 1$ and Theorem 3.1 yields the following result.

Corollary 6.1. *Let q be univalent in the unit disk $\triangle$, $0 \neq \beta \in \mathcal{C}$, and $0 < \mu < 1$. Suppose that q satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\mu}{\beta}\right)\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1+2\beta)\left(\frac{z}{\mathcal{P}^\alpha f(z)}\right)^\mu - 2\beta\left(\frac{\mathcal{P}^{\alpha-1} f(z)}{\mathcal{P}^\alpha f(z)}\right)\left(\frac{z}{\mathcal{P}^\alpha f(z)}\right)^\mu \prec q(z) + \frac{\beta}{\mu} zq'(z)$$

then

$$\left(\frac{z}{\mathcal{P}^\alpha f(z)}\right)^\mu \prec q(z),$$

and q is the best dominant.

Taking

$$\lambda_\alpha(t) = \binom{\alpha+\beta}{\alpha} \alpha\,(1-t)^{\alpha-1} t^\beta \quad (\alpha > 0; \beta > -1)$$

in (1.10), we obtain the integral operator Q^α_β given by

$$Q^\alpha_\beta f(z) = \binom{\alpha+\beta}{\alpha} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt \quad (f \in \mathcal{A}; \alpha > 0; \beta > -1),$$

where, as usual,

$$\binom{\kappa}{\nu} := \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\nu+1)\Gamma(\nu+1)} = \binom{\kappa}{\kappa-\nu} \quad (\kappa, \nu \in \mathbb{C})$$

in terms of familiar Gamma functions. The integral operator $Q^{\alpha}_{\beta}f(z)$ is equivalent to the convolution operator $\mathcal{L}(a,c)$ of Carlson and Shaffer [3].

In the case when

$$\alpha > 1, \quad \beta > -1, \quad \text{and} \quad 0 < \alpha + \beta \leq 3,$$

the conditions (1.11) and (1.12) are satisfied for $c = \alpha + \beta - 1$. From Theorem 3.1, we get the following result.

Corollary 6.2. *Let q be univalent in the unit disk $\triangle$, $0 \neq \beta_1 \in \mathbb{C}$, and $0 < \mu < 1$. Suppose that q satisfies the condition*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \ -\Re\left(\frac{\mu}{\beta_1}\right)\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1+\beta_1+\beta_1\beta)\left(\frac{z}{Q^{\alpha}_{\beta}f(z)}\right)^{\mu} - \beta_1(1+\beta)\left(\frac{Q^{\alpha-1}_{\beta}f(z)}{Q^{\alpha}_{\beta}f(z)}\right)\left(\frac{z}{Q^{\alpha}_{\beta}f(z)}\right)^{\mu} \prec q(z) + \frac{\beta}{\mu}zq'(z),$$

then

$$\left(\frac{z}{Q^{\alpha}_{\beta}f(z)}\right)^{\mu} \prec q(z),$$

and q is the best dominant.

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