Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **7** (2011), No. 2

SOLVING FRACTIONAL FORNBERG-WHITHAM EQUATION BY HOMOTOPY ANALYSIS METHOD

HASAN SABERI NIK AND REZA BUZHABADI

Abstract

In this paper, The analytical solution of the nonlinear Fornberg-Whitham equation with fractional time derivative was derived by means of the homotopy analysis method. The fractional derivatives are described in the Caputo sense. By choosing different values of the parameters in general formal numerical solutions, as a result, a very rapidly convergent series solution is obtained. The results reveal that the proposed method is very effective and simple for solving approximate solutions.

Mathematics Subject Classification 2000: MSC2010 database 65Z05 , ,
General Terms: Documentation, Languages
Additional Key Words and Phrases: Homotopy analysis method (HAM), Partial differential equation, FornbergWhitham equation, Caputo fractional derivative

1. INTRODUCTION

Interest in the concept of differentiation and integration to noninteger order has existed since the development of the classical calculus [Caputo 1967;Oldham 1974;Blank 1996]. By implication, mathematical modeling of many physical systems are governed by linear and nonlinear fractional differential equations in various applications in fluid mechanics, viscoelasticity, chemistry, physics, biology and engineering.
Since many fractional differential equations are nonlinear and do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. The Adomain decomposition method (ADM)[Momani 2006], the homotopy perturbation method (HPM)[Odibat 2008], the variational iteration method (VIM)[Odibat 2006] and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small.
In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), Liao [1992; 1997; 2003; 2004; 2005]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [Hayat 2005], the KdV-type equations [Abbasbandy 2008], finance problems [Zhu 2006], and so on. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series so-

lution.
The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.
In this paper, we have to solve the nonlinear time-fractional Fornberg-Whitham equation by the homotopy analysis method. This equation can be written in operator form as

$$u_t^\alpha - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{1}$$

subject to the initial condition

$$u(x,0) = e^{\frac{x}{2}} \tag{2}$$

The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 applying HAM for nonlinear time-fractional Fornberg-Whitham equation. Discussion and conclusions are presented in Section 5.

2. *D*ESCRIPTION ON THE FRACTIONAL CALCULUS

2.1 Definition

A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathrm{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in (0, \infty)$, and it is said to be in the space C_n^μ l if and only if $h(n) \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\nu$ if $\nu \leq \mu$.

2.2 Definition

The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad \alpha > 0, x > 0.) \tag{3}$$

$$J^0 f(x) = f(x).$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:
For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$
$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$
$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

2.3 Definition

For the concept of fractional derivative, there exist many mathematical definitions [West 2003; Miller 1993; Samko 1993; Caputo 1967; Podlubny 1999]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [Caputo 1967] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{4}$$

Here, we also need two basic properties about them:

$$\begin{aligned} &D^\alpha J^\alpha f(x) = f(x), \\ &J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0. \end{aligned} \tag{5}$$

2.4 Definition

The Mittag-Leffler function $E_\alpha(z)$ with $a > 0$ is defined by the following series representation, valid in the whole complex plane :

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, z \in C \tag{6}$$

3. BASIC IDEA OF HAM

To describe the basic ideas of the HAM, We consider the following differential equation :

$$D_t^\alpha u = 0, \qquad t > 0, \tag{7}$$

where the operator D_t^α stand for the fractional derivative and is defined as in Eq. (4), t denote an independent operator and $u(t)$ is an unknown function.
By means of generalizing the traditional homotopy method, Liao [Liao 1992] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(t;q) - u_0(t)] = q\, hH(t) D_t^\alpha[\phi(t;q)], \tag{8}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(t)$ is initial guesse of $u(t)$, $u(t;q)$ is unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t;0) = u_0(t), \phi(t;1) = u(t),$$

respectively. Thus, as q increases from 0 to 1, the solution $u(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $u(t;q)$ in Taylor series with respect to q, we have

$$\phi(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m, \tag{9}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{10}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (9) converges at $q = 1$, then we have

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t), \tag{11}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [Liao 2003]. As $h = -1$ and $H(t) = 1$, Eq. (8) becomes

$$(1-q)L[\phi_1(t;q) - u_0(t)] + q\ N[\phi_1(t;q)] = 0, \tag{12}$$

which is used mostly in the homotopy perturbation method [He 1999], where as the solution obtained directly, without using Taylor series . According to the definition (10), the governing equation can be deduced from the zero-order deformation equation (8). Define the vector

$$\vec{u}_n = \{u_0(t), u_1(t), \ldots, u_n(t)\},$$

Differentiating equation (8) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = h\,H(t)R_m(\vec{u}_{m-1}), \tag{13}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} D_t^{\alpha}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{14}$$

and Applying the RiemannLiouville integral operator J^{α} on both side of Eq. (13), we have

$$u_m(t) = \chi_m u_{m-1}(t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^+)\frac{t^i}{i!} + hH(t)J^{\alpha}R_m(\vec{u}_{m-1}), \tag{15}$$

It should be emphasized that $u_m(t)$ for $m \geq 1$ is governed by the linear equation (13) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work.
Liao [2003] proved that, as long as a series solution given by the homotopy analysis method converges, it must be one of exact solutions. So, it is important to ensure that the solution series is convergent. Note that the solution series contain the auxiliary parameter h, which we can choose properly by plotting the so-called h-curves to ensure solution series converge.

Remark 1. The parameter α, can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameter is bigger than 1, we will need more initial and boundary conditions such as $u_0'(x,0), u_0''(x,0), \cdots$ and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameter to $[0,1]$ to discuss.

4. SOLUTION OF THE PROBLEM

In order to illustrate the method discussed above, We first consider the following time-fractional Fornberg-Whitham equation

$$u_t^\alpha - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{16}$$

with initial condition

$$u(x,0) = e^{\frac{x}{2}} \tag{17}$$

For application of homotopy analysis method, in view of Eq. (16) and the initial condition given in Eq. (17), it in convenient to choose

$$u_0(x,t) = e^{\frac{x}{2}}, \tag{18}$$

as the initial approximate of Eq. (16). We choose the linear operator

$$L[\phi(x;q)] = D_t^\alpha[\phi(x;q)], \tag{19}$$

with the property $L(c) = 0$ where c is constant of integration. Furthermore, we define nonlinear operators as

$$\begin{aligned} N[\phi(x,t;q)] = D_t^\alpha[\phi(x,t;q)] - \phi_{xxt}(x,t;q) + \phi_x(x,t;q) - \phi(x,t;q)\phi_{xxx}(x,t;q) \\ +\phi(x,t;q)\phi_x(x,t;q) - 3\phi_x(x,t;q)\phi_{xx}(x,t;q), \end{aligned}$$

We construct the zeroth-order and the mth-order deformation equations where

$$\begin{aligned} R_m(\vec{u}_{m-1}) = D_t^\alpha(u_{m-1}) - (u_{m-1})_{xxt} + (u_{m-1})_x - \sum_{k=0}^{m-1} u_k(u_{m-1-k})_{xxx} \\ + \sum_{k=0}^{m-1} u_k(u_{m-1-k})_x - 3\sum_{k=0}^{m-1}(u_{m-1})_k(u_{m-1-k})_{xx}, \end{aligned}$$

We start with an initial approximation $u(x,0) = e^{\frac{x}{2}}$, thus we can obtain directly the other components as:

$$\begin{aligned} u_1 &= \frac{1}{2}\frac{ht^\alpha e^{\frac{x}{2}}}{\Gamma(\alpha+1)}, \\ u_2 &= \frac{-1}{4}\frac{e^{\frac{x}{2}}}{\alpha\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})}[4\alpha\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})h - 2t^\alpha h\Gamma(\alpha+\frac{1}{2}) - 2t^\alpha h^2\Gamma(\alpha+\frac{1}{2}) \\ &\quad +4^{-\alpha}h^2\pi^{\frac{1}{2}}t^{2\alpha-1}\alpha - 4^{-\alpha}h^2\pi^{\frac{1}{2}}t^{2\alpha}] \\ &\vdots \end{aligned}$$

When $h = -1$, it is easily seen that the solutions above are exactly the solutions in [Gupta 2011].

4.1 Results and discussion

In this subsection we first plot exat solution of the fractional Fornberg-Whitham equation, Figure 1 and Figure 2, show the HAM solutions with $h = -1, \alpha = 1$. Figure 3, shows the HAM solutions with $h = -1, \alpha = \frac{2}{3}$. Figure 4, shows the HAM solutions with $h = -1, \alpha = \frac{3}{4}$. The numerical results $u(x, t)$ for different t and α at $x = 1$ can be seen in Figure 5. Figure 2, clearly shows that, when $\alpha = 1$, the solution is very near to the exact solution.

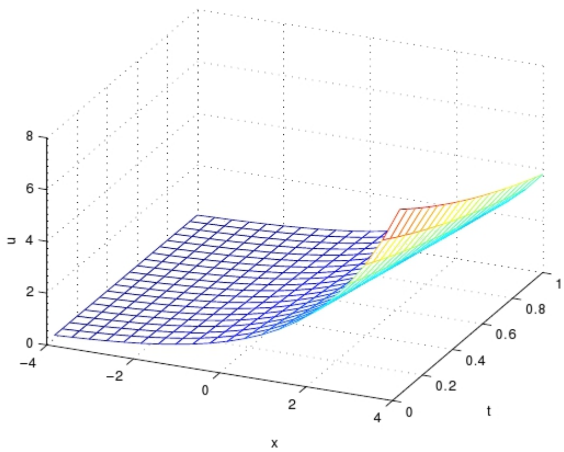

Fig. 1. Exat Solution of the fractional Fornberg-Whitham equation.

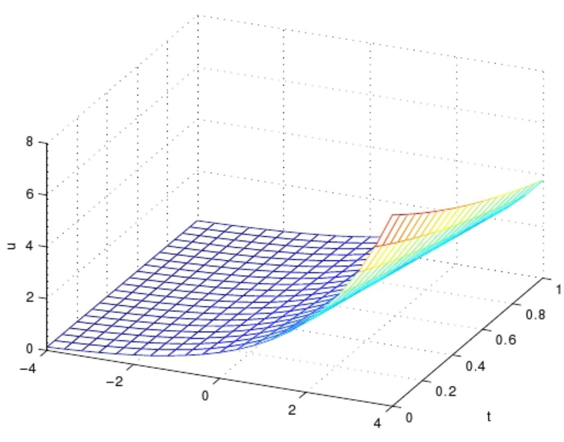

Fig. 2. Explicit numerical solutions with $h = -1, \alpha = 1$.

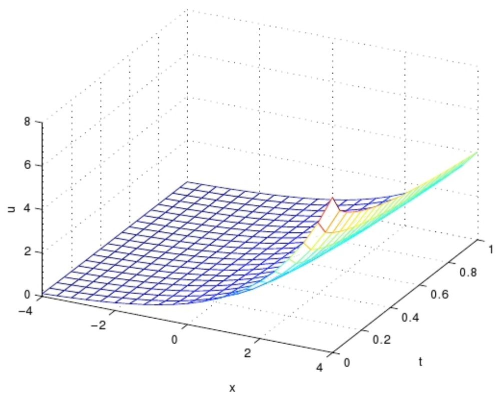

Fig. 3. Explicit numerical solutions with $h=-1, \alpha=\frac{2}{3}$.

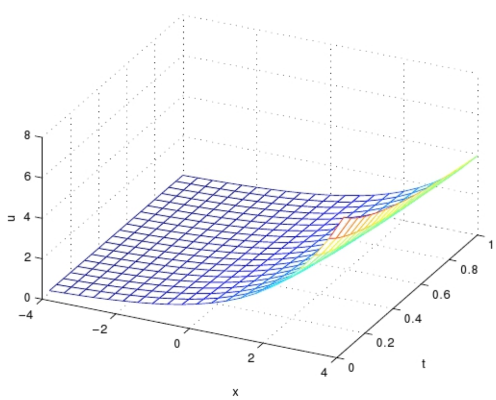

Fig. 4. Explicit numerical solutions with $h=-1, \alpha=\frac{3}{4}$.

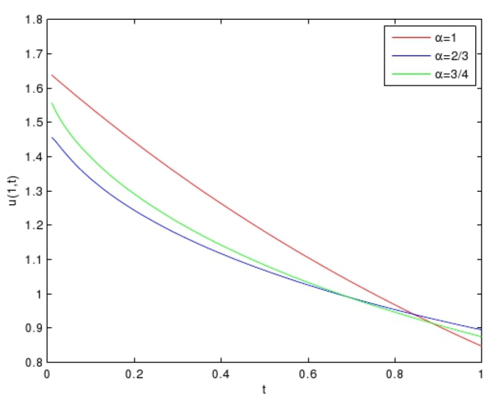

Fig. 5. Plots of 6-order HAM solution at $x=1$ for different values of α.

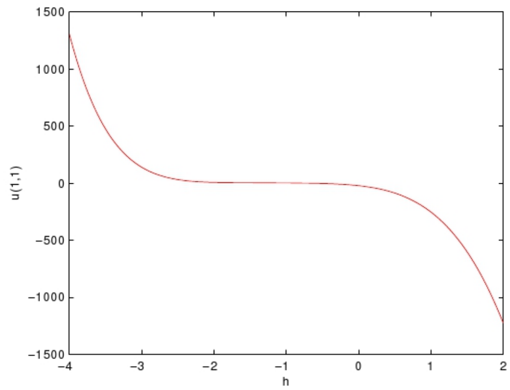

Fig. 6. The h-curves obtained from the 6-order HAM approximate solution.

As suggested by Liao [2003], the appropriate region for h is a horizontal line segment. We can investigate the influence of h on the convergence of, by plotting the curve of it versus h, as shown in Figure 6.

Remark 2. This example has been solved using homotopy perturbation method [Gupta 2011]. The graphs drawn by $h = -1$ are in excellent agreement with that graphs drawn with HPM.

5. CONCLUSION

In this paper, based on the symbolic computation Matlab, the HAM is directly extended to derive explicit and numerical solutions of the fractional Fornberg-Whitham equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting h, which is a fundamental qualitative difference in analysis between HAM and other methods. The obtained results demonstrate the reliability of the HAM and its wider applicability to fractional differential equation. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems. Matlab has been used for computations in this paper.

REFERENCES

Abbasbandy, S. 2008. Soliton solutions for the 5th-order KdV equation with the homotopy analysis method. *Nonlinear Dyn 51.* 83-87.

Blank, L. 1996. Numerical treatment of differential equations of fractional order. *Tech. Rep. Numerical Analysis Report, 287, The University of Manchester, Manchester, UK.*

Caputo, M. 1967. Linear models of dissipation whose Q is almost frequency independent. *Part II. J. Roy Austral Soc. vol. 13.* 529-539.

Gupta, P. K., Singh, M. 2011. Homotopy perturbation method for fractional Fornberg-Whitham equation. *Computers and Mathematics with Applications 61* 250-254.

Hayat, T., Khan, M., and Ayub, M. 2005. On non-linear flows with slip boundary condition. *ZAMP,56* 1012-1029.

He, J. H. 1999. Homotopy perturbation technique. *Comput. Meth. Appl. Eng., vol. 178.* 257-262.

Liao, S. J. 1992. The proposed homotopy analysis technique for the solution of nonlinear problems. *Ph.D. Thesis, Shanghai Jiao Tong University.*

Liao, S. J. 2003. Beyond perturbation: introduction to the homotopy analysis method.*CRC Press, Boca Raton: Chapman & Hall.*

Liao, S. J. 2004. On the homotopy anaylsis method for nonlinear problems.*Appl. Math. Comput. 147.* 499-513.

Liao, S. J. 2005. Comparison between the homotopy analysis method and homotopy perturbation method.*Appl. Math. Comput. 169.* 1186-1194.

Liao, S. J. 1997. Homotopy Analysis Method: A New Analytical Technique for Nonlinear Problems.*Journal of Commun Nonlinear Sci Numer Simul. 2-2.* 95-100.

Miller, K.S., Ross, B. 1993. An Introduction to the Fractional Calculus and Fractional Differential Equations.*Wiley, NewYork.*

Momani, S., Shawagfeh, N. T. 2006. Decomposition method for solving fractional Riccati differential equations.*Appl. Math. Comput, vol. 182.* 1083-1092.

Odibat, Z., Momani, S. 2008. Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order.*Chaos, Solitons and Fractals, vol. 36* 167-174.

Odibat, Z., Momani, S. 2006. Application of variation iteration method to nonlinear differential equations of fractional order.*Int. J . Nonlin. Sci. Numer. Simulat. vol. 1* 15-27.

Oldham, K. B., Spanier, J. 1974. The Fractional calculus.*New York: Academic Press.*

Podlubny, I. 1999. Fractional Differential Equations.*Academic Press, San Diego.*

Samko, S. G., Kilbas, A. A., and Marichev, O. I. 1993. Fractional Integrals and Derivatives.*Theory and Applications, Gordon and Breach, Yverdon.*

West, B. J., Bolognab, M., and Grigolini, P. 2003. Physics of Fractal Operators.*Springer, New York.*

Zhu, S. P. 2006. An exact and explicit solution for the valuation of American put options.*Quantitative Finance. 6* 229-242.

HASAN SABERI NIK,
Islamic Azad university,
Neyshabour branch,
Neyshabour, Iran
e-mail: hssn@yahoo.com

REZA BUZHABADI,
Islamic Azad university,
Neyshabour branch,
Neyshabour, Iran
e-mail: bozhabadi@yahoo.com

Received March 2011