Journal of the Applied Mathematics, Statistics and Informatics (JAMSI), **7** (2011), No. 2

# GENERALIZED FRACTIONAL DIFFERENTIATION OF THE MULTIVARIABLE H-FUNCTION

KANTESH GUPTA AND ALPANA GUPTA


### Abstract

In the present paper, we study and develop the generalized fractional differential operators introduced by Saigo [6]. First, we establish two theorems that give the images of the multivariable H – function in Saigo operators. On account of the general nature of the Saigo operators, H-function of two variables and also of the multivariable H – function, a large number of new and known theorems involving Riemann – Liouville, Erdélyi – Kober fractional differential operators and several special functions notably generalized Wright hypergeometric function, Mittag –Leffler function , Whittaker function and Bessel function follow as special cases of our main findings. The important results obtained by Kilbas [1], Kilbas and Sebastian [3] and Saxena, Ram & Suthar [7] follow as special cases of our results.


**Mathematics Subject Classification 2000**: 26A33, 33C60, 33E12
**Additional Key Words and Phrases**: Fractional differential operators by Saigo, Riemann- Liouville and Erdélyi – Kober, **H**- function, multivariable **H**- function, Mittag– Leffler function.

## 1. INTRODUCTION

In this paper, we shall study and develop the following generalized fractional differential operators introduced by Saigo [6].

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,\,-\beta-n,\,\alpha+\eta-n} f\right)(x) \quad (\mathrm{Re}(\alpha)\geq 0,\ n=[\mathrm{Re}(\alpha)]+1) \tag{1}$$

and

$$\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) = \left(\frac{-d}{dx}\right)^n \left(I_{-}^{-\alpha+n,\,-\beta-n,\,\alpha+\eta} f\right)(x) \quad (\mathrm{Re}(\alpha)\geq 0,\ n=[\mathrm{Re}(\alpha)]+1) \tag{2}$$

where $\alpha,\beta,\eta\in C, \mathrm{Re}(\alpha)\geq 0$ and $I_{0+}^{\alpha,\beta,\eta}, I_{-}^{\alpha,\beta,\eta}$ known as generalized fractional differential operators introduced by Saigo [6].

when $\beta=-\alpha$, the above operators (1) and (2) reduce to the following classical Riemann – Liouville fractional differential operators of order $\alpha\in C$ $(\mathrm{Re}(\alpha)\geq 0)$ [4, p.80, Eqs.(2.2.3), (2.2.4) ]:

$$(D_{0+}^{\alpha,-\alpha,\eta} f)(x) = \left(D_{0+}^{\alpha} f\right)(x)$$

$$\equiv \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \qquad (x>0, n=[\mathrm{Re}(\alpha)]+1) \tag{3}$$

and
$$(D_{-}^{\alpha,-\alpha,\eta} f)(x) = \left(D_{-}^{\alpha} f\right)(x)$$

$$\equiv \left(\frac{-d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^{\infty} \frac{f(t)dt}{(t-x)^{\alpha-n+1}}, \qquad (x>0,\ n=[\mathrm{Re}(\alpha)]+1) \tag{4}$$

Again, if $\beta = 0$ Equations (1) and (2) reduce to the Erdélyi – Kober fractional differential operators defined below [ 4, p.109, Eqs.(2.6.35), (2.6.36) ]:

$$\left(D_{0+}^{\alpha,0,\eta} f\right)(x) = \left(D_{\eta,\alpha}^{+} f\right)(x)$$

$$\equiv x^{-\eta}\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^{x} \frac{t^{\alpha+\eta} f(t)dt}{(x-t)^{\alpha-n+1}}, \qquad (x>0,\ n=[\mathrm{Re}(\alpha)]+1) \tag{5}$$

and
$$\left(D_{-}^{\alpha,0,\eta} f\right)(x) = \left(D_{\eta,\alpha}^{-} f\right)(x)$$

$$\equiv x^{\alpha+\eta}\left(\frac{-d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^{\infty} \frac{t^{-\eta} f(t)dt}{(t-x)^{\alpha-n+1}}, \qquad (x>0,\ n=[\mathrm{Re}(\alpha)]+1) \tag{6}$$

The H-function of *r* complex variables $z_1, z_2, \ldots z_r$ was introduced by Srivastava and Panda [9]. In this paper we shall define and represent it in the following form [8, p.251, Eq. (c.1)] :

$$H[z_1,\ldots,z_r] = H_{P,Q\,:P_1,Q_1\,;\ldots;\,P_r,Q_r}^{0,N:M_1,N_1\,;\ldots;\,M_r,N_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{l} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \psi\left(\xi_1,\ldots,\xi_r\right)\prod_{i=1}^{r}\left\{\phi_i(\xi_i)\, z_i^{\xi_i}\right\} d\xi_1 \ldots d\xi_r \tag{7}$$

where $\omega = \sqrt{(-1)}$ and

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i}\Gamma\left(d_j^{(i)} - \delta_j^{(i)}\xi_i\right)\prod_{j=1}^{N_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right)}{\prod_{j=M_i+1}^{Q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)\prod_{j=N_i+1}^{P_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)} \quad \forall\, i \in (1,\ldots,r) \tag{8}$$

$$\psi(\xi_1,....\xi_r)=\frac{\prod_{j=1}^{N}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=N+1}^{P}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)\prod_{j=1}^{Q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)} \tag{9}$$

The nature of contours $L_1,...,L_r$ in (7), the various special cases and other details of the above function can be found in the book referred to above. Also, it is assumed throughout the present work that this function always satisfy the appropriate existence and convergence conditions of its defining integral [8, pp.252-253, Eqs. (c.4-c.6)].

## 2. PRELIMINARY RESULTS

The following lemmas will be required to establish our main results.

**Lemma 1** [ 3, p.327, Eq.(22) to (25) ]: Let $\alpha,\beta,\eta\in C$ be such that

$$\operatorname{Re}(\alpha)\geq 0, \quad \operatorname{Re}(\sigma)>-\min[0, \operatorname{Re}(\alpha+\beta+\eta)] \tag{10}$$

Then there holds the relation

$$(D_{0+}^{\alpha,\beta,\eta}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)}x^{\sigma+\beta-1} \qquad (x>0) \tag{11}$$

In particular, for x>0

$$(D_{0+}^{\alpha}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}x^{\sigma-\alpha-1}, \qquad (\operatorname{Re}(\alpha)\geq 0, \operatorname{Re}(\sigma)>0) \tag{12}$$

$$(D_{\eta,\alpha}^{+}t^{\sigma-1})(x)=\frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)}x^{\sigma-1}, \ (\operatorname{Re}(\alpha)\geq 0, \operatorname{Re}(\sigma)>-\operatorname{Re}(\alpha+\eta)) \tag{13}$$

**Lemma** 2 [ 3, p.328, Eq.(26) to (29) ]: Let $\alpha,\beta,\eta\in C$ be such that

$$\operatorname{Re}(\alpha)\geq 0, \quad \operatorname{Re}(\sigma)<1+\min[\operatorname{Re}(-\beta-n), \operatorname{Re}(\alpha+\eta)], \ n=[\operatorname{Re}(\alpha)]+1 \tag{14}$$

Then

$$(D_{-}^{\alpha,\beta,\eta}t^{\sigma-1})(x)=\frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\beta)}x^{\sigma+\beta-1} \qquad (x>0) \tag{15}$$

In particular, for x>0

$$(D_{-}^{\alpha}t^{\sigma-1})(x)=\frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)}x^{\sigma-\alpha-1}, \quad (\operatorname{Re}(\alpha)\geq 0, \operatorname{Re}(\sigma)<1+\operatorname{Re}(\alpha)-n) \tag{16}$$

$$(D_{\eta,\alpha}^{-}t^{\sigma-1})(x)=\frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta)}x^{\sigma-1} \tag{17}$$

$$(\operatorname{Re}(\alpha)\geq 0, \quad \operatorname{Re}(\sigma)<1+\operatorname{Re}(\alpha+\eta)-n)$$

## 3. MAIN RESULTS

**Theorem 1** :

$$\left\{ D_{0+}^{\alpha,\beta,\eta}\left( t^{\sigma-1} H_{P,Q:P_1,Q_1;\ldots;P_r,Q_r}^{0,N:M_1,N_1;\ldots;M_r,N_r}\begin{bmatrix} z_1 t^{\rho_1} \\ \vdots \\ z_r t^{\rho_r}\end{bmatrix}\right)\right\}(x)$$

$$= x^{\sigma+\beta-1} H_{P+2,Q+2:P_1,Q_1;\ldots\ldots;P_r,Q_r}^{0,N+2:M_1,N_1;\ldots\ldots;M_r,N_r}\left[\begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r}\end{matrix}\left|\begin{matrix} A^*:C^* \\ B^*:D^*\end{matrix}\right.\right] \quad (18)$$

where

$$A^* = (1-\sigma;\rho_1,..,\rho_r),(1-\sigma-\alpha-\beta-\eta;\rho_1,..,\rho_r),\left(a_j;\alpha_j^{(1)},..,\alpha_j^{(r)}\right)_{1,P}$$

$$B^* = (1-\sigma-\beta;\rho_1,..,\rho_r),(1-\sigma-\eta;\rho_1,..,\rho_r),\left(b_j;\beta_j^{(1)},..,\beta_j^{(r)}\right)_{1,Q}$$

$$C^* = \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}$$

$$D^* = \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}$$

The sufficient conditions of validity of (18) are

(i) $\alpha,\beta,\eta,\sigma,z_i \in C$ and $\rho_i > 0\ \forall i \in \{1,2\ldots,r\}$

(ii) $|\arg z_i| < \frac{1}{2}\Omega_i\pi$ and $\Omega_i > 0$

where

$$\Omega_i = -\sum_{j=N+1}^{P}\alpha_j^{(i)} - \sum_{j=1}^{Q}\beta_j^{(i)} + \sum_{j=1}^{N_i}\gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i}\gamma_j^{(i)} + \sum_{j=1}^{M_i}\delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i}\delta_j^{(i)} > 0, \forall i \in \{1,2,..,r\}$$

(iii) $\operatorname{Re}(\alpha) \geq 0$ and

$$\operatorname{Re}(\sigma) + \sum_{i=1}^{r}\rho_i \min_{1\le j\le M_i} \operatorname{Re}\left( d_j^{(i)} \big/ \delta_j^{(i)} \right) > -\min[0, \operatorname{Re}(\alpha+\beta+\eta)]$$

**Proof** : In order to prove theorem 1, we first express the multivariable H-function occurring in left – hand side of (18) in terms of Mellin-Barnes contour integral with the help of equation (7) and interchange the order of integration (which is permissible under the conditions stated), it takes the following form (say $I_1$) after a little simplification

$$I_1 = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1,..,\psi_r) \prod_{i=1}^{r} \left\{ \phi_i(\xi_i) z_i^{\xi_i} \right\} d\xi_{1,...,} d\xi_r \left( D_{0+}^{\alpha,\beta,\eta} t^{\sigma+\rho_1\xi_1+....\rho_r\xi_r-1} \right)(x)$$

Finally, applying the lemma 1 representing by equation (11) and reinterpreting the Mellin-Barnes contour integral thus obtained in terms of the H-function of N+2 variables, we arrive at the right hand side of (18) after a little simplification.

If we put $\beta = -\alpha$ in theorem 1, we arrive at the following new and interesting corollary concerning Riemann – Liouville fractional differential operators.

**Corollary 1.1** :

$$\left\{ D_{0+}^{\alpha} \left( t^{\sigma-1} H_{P,Q:P_1,Q_1;...;P_r,Q_r}^{0,N:M_1,N_1;...;M_r,N_r} \begin{bmatrix} z_1 t^{\rho_1} \\ \vdots \\ z_r t^{\rho_r} \end{bmatrix} \right) \right\}(x)$$

$$= x^{\sigma-\alpha-1} H_{P+1,Q+1:P_1,Q_1;......;P_r,Q_r}^{0,N+1:M_1,N_1;.....;M_r,N_r} \left[ \begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{matrix} \middle| \begin{matrix} A_1^* : C^* \\ B_1^* : D^* \end{matrix} \right] \tag{19}$$

where

$$A_1^* = (1-\sigma; \rho_1,...,\rho_r), \left(a_j; \alpha_j^{(1)},....,\alpha_j^{(r)}\right)_{1,P}$$

$$B_1^* = (1-\sigma+\alpha; \rho_1,..,\rho_r), \left(b_j; \beta_j^{(1)},..,\beta_j^{(r)}\right)_{1,Q}$$

$C^*$ and $D^*$ are same as given in (18) and the conditions of existence of the above corollary follow easily with the help of theorem 1.

Again, if we put $\beta = 0$ in theorem 1, we get the following result which is also believed to be new and pertains to Erdélyi – Kober fractional differential operators.

**Corollary 1.2** :

$$\left\{ D_{\eta,\alpha}^{+} \left( t^{\sigma-1} H_{P,Q:P_1,Q_1;...;P_r,Q_r}^{0,N:M_1,N_1;...;M_r,N_r} \begin{bmatrix} z_1 t^{\rho_1} \\ \vdots \\ z_r t^{\rho_r} \end{bmatrix} \right) \right\}(x)$$

$$= x^{\sigma-1} H_{P+1,Q+1:P_1,Q_1;......;P_r,Q_r}^{0,N+1:M_1,N_1;.....;M_r,N_r} \left[ \begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{matrix} \middle| \begin{matrix} A_2^* : C^* \\ B_2^* : D^* \end{matrix} \right] \tag{20}$$

where

$$A_2^* = (1-\sigma-\alpha-\eta; \rho_1,..,\rho_r), \left(a_j; \alpha_j^{(1)},..,\alpha_j^{(r)}\right)_{1,P}$$

$$B_2^* = (1-\sigma-\eta; \rho_1,..,\rho_r), \left(b_j; \beta_j^{(1)},..,\beta_j^{(r)}\right)_{1,Q}$$

$C^*$ and $D^*$ are same as mentioned in (18) and the conditions of validity of aforementioned corollary can be easily obtained from the existence conditions given with the main theorem.

**Theorem 2** :

$$\left\{D_{-}^{\alpha,\beta,\eta}\left(t^{\sigma-1}H_{P,Q:P_1,Q_1;\ldots;P_r,Q_r}^{0,N:M_1,N_1;\ldots;M_r,N_r}\begin{bmatrix} z_1 t^{-\rho_1} \\ \vdots \\ z_r t^{-\rho_r}\end{bmatrix}\right)\right\}(x)$$

$$= x^{\sigma+\beta-1}H_{P+2,Q+2:P_1,Q_1;\ldots\ldots;P_r,Q_r}^{0,N+2:M_1,N_1;\ldots\ldots;M_r,N_r}\left[\begin{matrix} z_1 x^{-\rho_1} \\ \vdots \\ z_r x^{-\rho_r}\end{matrix}\middle| \begin{matrix} A^{**}:C^* \\ B^{**}:D^*\end{matrix}\right] \tag{21}$$

where

$$A^{**}=(\sigma+\beta;\rho_1,..,\rho_r),(\sigma-\alpha-\eta;\rho_1,..,\rho_r),\left(a_j;\alpha_j^{(1)},..,\alpha_j^{(r)}\right)_{1,P}$$

$$B^{**}=(\sigma;\rho_1,..,\rho_r),(\sigma+\beta-\eta;\rho_1,..,\rho_r),\left(b_j;\beta_j^{(1)},..,\beta_j^{(r)}\right)_{1,Q}$$

$C^*$ and $D^*$ are same as defined in (18) and provided that the following conditions are satisfied :

(i) $\text{Re}(\alpha)\geq 0$ and

$$\text{Re}(\sigma)-\sum_{i=1}^{r}\rho_i \min_{1\leq j\leq M_i}\text{Re}\left(d_j^{(i)}\middle/\delta_j^{(i)}\right)<1+\min[\text{Re}(-\beta-n),\text{Re}(\alpha+\eta)]$$

(ii) and the conditions (i) and (ii) given with the theorem 1 are also satisfied.

**Proof** : The proof of theorem 2 can be develop on the lines similar to those given with theorem 1 with the help of lemma 2.

If we put $\beta=-\alpha$ and $\beta=0$ in theorem 2, in succession we shall easily arrive at the corresponding corollaries concerning Riemann – Liouville and Erdélyi –Kober fractional differential operators respectively. We omit the details.

## 4. SPECIAL CASES OF THEOREM 1

**(i)** If we reduce the multivariable H-function involved in (18) to the product of r different Whittaker functions [8, p.18, Eq.(2.6.7)] and taking $\rho_i$ =1, we arrive at the following new and interesting result after a little simplification :

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}\prod_{i=1}^{r}e^{\frac{-z_i t}{2}}W_{\lambda_i,\mu_i}(z_i t)\right)\right\}(x)$$

$$= x^{\sigma+\beta-1}H_{2,2:1,2;\ldots\ldots;1,2}^{0,2:2,0;\ldots\ldots;2,0}\left[\begin{matrix} z_1 x \\ \vdots \\ z_r x\end{matrix}\middle| \begin{matrix} A_3^*:C_3^* \\ B_3^*:D_3^*\end{matrix}\right] \tag{22}$$

where

$$A_3^* = \left(1-\sigma;\underbrace{1,...,1}_{r}\right),\left(1-\sigma-\alpha-\beta-\eta;\underbrace{1,...,1}_{r}\right)$$

$$B_3^* = \left(1-\sigma-\beta;\underbrace{1,...,1}_{r}\right),\left(1-\sigma-\eta;\underbrace{1,...,1}_{r}\right)$$

$$C_3^* = (1-\lambda_1,1);...;(1-\lambda_r,1)$$

$$D_3^* = \left(\frac{1}{2}\pm\mu_1,1\right);...;\left(\frac{1}{2}\pm\mu_r,1\right)$$

The conditions of validity of (22) can be easily derived from those of (18).

**(ii)** If we reduce the multivariable H-function into the product of two Fox H-function and then reduce one H-function to the exponential function by taking $\rho_1=1$, we get the following result after a little simplification:

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}e^{-z_1 t}\quad H_{P_2,Q_2}^{M_2,N_2}\left[z_2 t^{\rho_2}\left|\begin{matrix}(c_j,\gamma_j)_{1,P_2}\\(d_j,\delta_j)_{1,Q_2}\end{matrix}\right.\right]\right)\right\}(x)$$
$$= x^{\sigma+\beta-1}H_{2,2:0,1;P_2,Q_2}^{0,2:1,0;M_2,N_2}\left[\begin{matrix}z_1 x\\z_2 x^{\rho_2}\end{matrix}\left|\begin{matrix}A_4^*:- \quad ;(c_j,\gamma_j)_{1,P_2}\\B_4^*:(0,1);(d_j,\delta_j)_{1,Q_2}\end{matrix}\right.\right] \tag{23}$$

where

$$A_4^* = (1-\sigma;1,\rho_2),(1-\sigma-\alpha-\beta-\eta;1,\rho_2)$$

$$B_4^* = (1-\sigma-\beta;1,\rho_2),(1-\sigma-\eta;1,\rho_2)$$

The conditions of validity of the above result easily follow from (18).

Further on letting $z_1 \to 0$ in the above equation, it takes the form:

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}H_{P_2,Q_2}^{M_2,N_2}\left[z_2 t^{\rho_2}\left|\begin{matrix}(c_j,\gamma_j)_{1,P_2}\\(d_j,\delta_j)_{1,Q_2}\end{matrix}\right.\right]\right)\right\}(x)$$
$$= x^{\sigma+\beta-1}H_{P_2+2,\,Q_2+2}^{M_2\;,\,N_2+2}\left[z_2 x^{\rho_2}\left|\begin{matrix}A_5^*\\B_5^*\end{matrix}\right.\right] \tag{24}$$

where

$$A_5^* = (c_j,\gamma_j)_{1,N_2},(1-\sigma,\rho_2),(1-\sigma-\alpha-\beta-\eta,\rho_2),(c_j,\gamma_j)_{N_2+1,P_2}$$

$$B_5^* = (d_j,\delta_j)_{1,Q_2},(1-\sigma-\beta,\rho_2),(1-\sigma-\eta,\rho_2)$$

The conditions of validity of the above result follow easily from the conditions given with theorem 1.

If we put $\beta=-\alpha$ and make suitable adjustments in the parameters, we arrive at a known result recorded in the book by Kilbas and Saigo [2,p.55,Eq.(2.7.22)].

Further, if we put $\beta=0$ in the equation (26), we shall easily arrive at the corresponding result concerning Erdélyi –Kober fractional differential operators respectively. We omit the details.

**(iii)** If we reduce the H-function to the generalized Wright hypergeometric function [8, p.19, Eq.(2.6.11) ] in the result given by (24), we get the following interesting result after a little simplification.

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}{}_P\psi_Q\left[z_2 t^{\rho_2}\middle|\begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]\right)\right\}(x)$$

$$= x^{\sigma+\beta-1}{}_{P+2}\psi_{Q+2}\left[z_2 x^{\rho_2}\middle|\begin{matrix}(c_j,\gamma_j)_{1,P},(\sigma,\rho_2),(\sigma+\alpha+\beta+\eta,\rho_2)\\(d_j,\delta_j)_{1,Q},(\sigma+\beta,\rho_2),(\sigma+\eta,\rho_2)\end{matrix}\right] \tag{25}$$

The conditions of validity of (25) can be easily obtained from those of (18).

If we put $\beta=-\alpha$ in the above result, we get a known important result given by Kilbas [1, p.119, Eq.(14) ].

Again if we put $\beta=0$ in the equation (25), it reduces to

$$\left\{D_{\eta,\alpha}^{+}\left(t^{\sigma-1}{}_P\psi_Q\left[z_2 t^{\rho_2}\middle|\begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]\right)\right\}(x)$$

$$= x^{\sigma-1}{}_{P+1}\psi_{Q+1}\left[z_2 x^{\rho_2}\middle|\begin{matrix}(c_j,\gamma_j)_{1,P},(\sigma+\alpha+\eta,\rho_2)\\(d_j,\delta_j)_{1,Q},(\sigma+\eta,\rho_2)\end{matrix}\right] \tag{26}$$

The conditions of existence of the above result follow easily with the help of (18).

**(iv)** Further, on taking $z_2=-1,\rho_2=1$ in the equation (24) and reducing the H-function occurring therein to generalized Mittag – Leffler function [5], we easily get after a little simplification the following result which is also believed to be new.

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left(t^{\sigma-1}E_{u,v}^{\upsilon}(t)\right)\right\}(x)$$

$$=\frac{x^{\sigma+\beta-1}}{\Gamma(\upsilon)}H_{3,4}^{1,3}\left[-x\middle|\begin{matrix}(1-\upsilon,1),(1-\sigma,1),(1-\sigma-\alpha-\beta-\eta,1)\\(0,1),(1-v,u),(1-\sigma-\beta,1),(1-\sigma-\eta,1),\end{matrix}\right] \tag{27}$$

The conditions of validity of the above result follow directly from those given with (18)

If we put $\beta=-\alpha$ in the above result, we get a known result due to Saxena et al. [7, p.170, Eq.(2.6) ].

If we put $\beta=0$ in the result given by (27), we arrive at the following result after a little simplification.

$$\left\{D_{\eta,\alpha}^{+}\left(t^{\sigma-1}E_{u,v}^{\upsilon}(t)\right)\right\}(x)$$

$$=\frac{x^{\sigma-1}}{\Gamma(\upsilon)}H_{2,3}^{1,2}\left[-x\middle|\begin{matrix}(1-\upsilon,1),(1-\sigma-\alpha-\eta,1)\\(0,1),(1-v,u),(1-\sigma-\eta,1)\end{matrix}\right] \tag{28}$$

The conditions of validity of (28) can be easily derived from those of (18).

**(v)** if we take $z_2 = \frac{\lambda^2}{4}, \rho_2 = 2$ and reduce the H-function to the Bessel function of first kind in the equation (24), we get known results obtained by Kilbas and Sebastian [3, p.330, Eq.(31) to (35)].

A number of several special cases of theorem 2 can also be obtained but we do not mention them here on account of lack of space.

## 5. REFERENCES

1. Kilbas, A.A. 2005, Fractional calculus of the generalized Wright function, Fract.Calc. Appl. Anal. 8(2), 113-126.
2. Kilbas, A.A. and Saigo, M. 2004, H-transforms.Theory and Applications, Chapman & Hall/CRC London, New York.
3. Kilbas, A.A. and Sebastian, N. 2008, Generalized fractional differentiation of Bessel function of the first kind, Mathematica Balkanica 22, 323-346.
4. Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. 2006, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam.
5. Prabhakar, T.R. 1971, A singular integral equation with a generalized Mittag – Leffler function in the kernel, Yokohama Math J. 19, 7-15.
6. Saigo, M. 1977/78, A remark on integral operators involving the gauss hypergeometric functions, Math. Rep. College of General Edu. Kyushu University, bf 11, 135-143.
7. Saxena, R.K., Ram, J. and Suthar, D.L. 2009, Fractional calculus of generalized Mittag – Leffler functions, J. Indian Acad. Math. 31(1), 165-172.
8. Srivastava, H.M., Gupta, K.C. and Goyal, S.P. 1982, The H- function of One and Two variables with Applications, South Asian Publications, New Delhi.
9. Srivastava, H.M. and Panda, R. 1976, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284, 265-274.

Kantesh Gupta
Deparment of Mathematics,
Malaviya National Institute of Technology,
Jaipur-302017
Email : kantesh_g@indiatimes.com

Alpana Gupta
Deparment of Mathematics,
Malaviya National Institute of Technology,
Jaipur-302017

Received July 2010